61. *An Extension of an Integral. II*

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1. Lemmas. This section is the continuation of section 3 in [1].

Assumption 3. *$\mathcal{I}$ is an abstract integral with respect to $(\mathcal{S}, \mathcal{G}, J)$.*

For each $f \in \mathcal{F}$, we can define a map μ_f of $\mathcal{R}(f)$ into J by $\mu_f(X) = \mathcal{I}(X, Xf)$ for $X \in \mathcal{R}(f)$.

Lemma 13. *The map μ_f is a J-valued pre-measure on $\mathcal{R}(f)$ for any $f \in \mathcal{F}$.*

Lemma 14. *If $f, g \in \mathcal{F}$ and $X \in \mathcal{R}(f) \cap \mathcal{R}(g)$, then $X \in \mathcal{R}(f+g)$ and $\mu_{f+g}(X) = \mu_f(X) + \mu_g(X)$.*

Lemma 15. *Suppose that $f \in \mathcal{F}, X \in \mathcal{S}$, and $Y \in \bar{\Sigma}$. Then $XY \in \mathcal{R}(f)$ if and only if $X \in \mathcal{R}(Yf)$, and these mutually equivalent conditions imply that $\mu_f(XY) = \mu_{Yf}(X)$.*

Proof. This follows from Lemma 7 in [1].

Let $\mathcal{V}$ be the system of neighbourhoods of $0 \in J$. Denote by Ω the set of all elements $(X, f) \in \tilde{\Omega}$ satisfying the following condition: for any $\xi, \eta \in \Sigma(f)$ such that $\bar{\xi} = \bar{\eta} = X$ and for any $V \in \mathcal{V}$, there exists a positive integer n such that $\mu_f(\xi(l)) - \mu_f(\eta(m)) \in V$ for any $l \geqq n$ and $m \geqq n$.

Lemma 16. *$(XY, f) \in \Omega$ if and only if $(X, Yf) \in \Omega$ for any $X, Y \in \bar{\Sigma}$ and $f \in \mathcal{F}$.*

Proof. Suppose that $(XY, f) \in \Omega$. Lemma 11 implies that $(X, Yf) \in \tilde{\Omega}$. Let ξ and η be elements of $\Sigma(Yf)$ such that $\bar{\xi} = \bar{\eta} = X$ and let V be an element of $\mathcal{V}$. It follows from Corollary to Lemma 7 that $Y\xi, Y\eta \in \Sigma(f)$ and $\overline{Y\xi} = \overline{Y\eta} = XY$. Hence we have an n such that $\mu_f((Y\xi)(l)) - \mu_f((Y\eta)(m)) \in V$ for any $l \geqq n$ and $m \geqq n$. For this n and for $l \geqq n$ and $m \geqq n$ we have $\mu_{Yf}(\xi(l)) - \mu_{Yf}(\eta(m)) = \mu_f(\xi(l)Y) - \mu_f(\eta(m)Y) = \mu_f((Y\xi)(l)) - \mu_f((Y\eta)(m)) \in V$. Thus we have $(X, Yf) \in \Omega$. Conversely suppose that $(X, Yf) \in \Omega$. $(XY, f) \in \tilde{\Omega}$ follows from Lemma 11. Let ζ_i be elements of $\Sigma(f)$ such that $\bar{\zeta}_i = XY$ for $i = 1, 2$, and let V be an element of $\mathcal{V}$. Lemma 8 implies that there are $\xi_i \in \Sigma(Yf)$ such that $\bar{\xi}_i = X$ and $\zeta_i = Y\xi_i$ for $i = 1, 2$. Since $(X, Yf) \in \Omega$, we have an n such that $\mu_{Yf}(\xi_1(l_1)) - \mu_{Yf}(\xi_2(l_2)) \in V$ for any $l_i \geqq n$. For this n and for $l_i \geqq n$, $i = 1, 2$, we have $\mu_f(\zeta_1(l_1)) - \mu_f(\zeta_2(l_2)) = \mu_f((Y\xi_1)(l_1)) - \mu_f((Y\xi_2)(l_2)) = \mu_f(\xi_1(l_1)Y) - \mu_f(\xi_2(l_2)Y) = \mu_{Yf}(\xi_1(l_1)) - \mu_{Yf}(\xi_2(l_2)) \in V$, which implies that $(XY, f) \in \Omega$. Thus the lemma is proved.