60. *An Extension of an Integral. I*

By Masahiro TAKAHASHI
Institute of Mathematics, College of General Education, Osaka University

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1. Introduction. An integral σ with respect to an integral structure Γ was defined in the author [3]. An example of integrals of this type is 1-dimensional (or generally n-dimensional) Lebesgue integral of bounded measurable functions over measure-finite measurable sets (see Introduction in [1]). In this case, however, we can not deal with such integrals as

$$\int_{-\infty}^{\infty} f(x)dx \qquad \text{where } f(x)=\begin{cases} x^{-2} & (1<x) \\ x^{-1/2} & (0<x\leqq 1) \\ 0 & (x\leqq 0) \end{cases}$$

in our way. We shall extend in this paper the integral σ to an 'integral' $\bar{\sigma}$ and then integrals of the above type may be dealt in terms of $\bar{\sigma}$.

2. Extension theorems. *Let $\Gamma=(\Lambda;\ \mathcal{S}, \mathcal{G}, Q)$ be an integral structure and σ an integral with respect to Γ.*

Denote by $\mathcal{M}, \mathcal{F}$, and J the total ring, the total functional group, and the third group, respectively, of Λ and let $\bar{\mathcal{S}}$ be the σ-ring generated by $\mathcal{S}$.

Let Ω be the set of all elements (X, f, μ) of $\mathcal{M}\times\mathcal{F}\times Q$ satisfying the following conditions:

1) There exist $X_i\in\mathcal{S}, i=1,2,\cdots$, such that $X_i f\in\mathcal{G}$ for any i and such that $X_i\uparrow X$ $(i\to\infty)$.

2) If $X_i^{(k)}\in\mathcal{S}, X_i^{(k)}f\in\mathcal{G}$, for $i=1,2,\cdots$, and if $X_i^{(k)}\uparrow X(i\to\infty)$, where $k=1,2$, then for any neighborhood V of $0\in J$ there exists a positive integer n such that $\sigma(X_l^{(1)}, X_l^{(1)}uf, \mu)-\sigma(X_m^{(2)}, X_m^{(2)}f, \mu)\in V$ for any $l\geqq n$ and $m\geqq n$.

The set Ω defined above will be called the *carrier* of Γ.

Let us assume the following:

1) $\sigma(X_i, g, \mu)\to 0$ $(i\to\infty)$ *for* $X_i\in\mathcal{S}, i=1,2,\cdots$, *such that* $X_i\downarrow 0$ $(i\to\infty)$, *for any* $g\in\mathcal{G}$ *and* $\mu\in Q$.

2) $\mathcal{S}$ *is a pseudo-σ-ring.*

3) J *is Hausdorff and complete.*

Then we have the following theorems, which will be proved in Part II of this paper.

Theorem 1. *Under the above assumptions,*

1) $\mathcal{S}\times\mathcal{G}\times Q\subset\Omega\subset\bar{\mathcal{S}}\times\mathcal{F}\times Q$.

2) *For any* $X, Y\in\bar{\mathcal{S}}, f\in\mathcal{F}$, *and* $\mu\in Q$, *it holds that* $(XY, f, \mu)\in\Omega$