85. *Remarks on Hypoellipticity of Degenerate Parabolic Differential Operators*

By Yoshio KATO

Department of Mathematics, Faculty of Engineering, Nagoya University

(Comm. by Kinjirô KUNUGI, M. J. A., April 12, 1971)

§1. Introduction. We have discussed in [2] the hypoellipticity of linear partial differential operators of the form

(1) $$P=\frac{\partial}{\partial t}+L(t,x;D_x),\qquad x=(x_1,\cdots,x_n)\in R^n,$$

where $D_x=(-i\partial/\partial x_1,\cdots,-i\partial/\partial x_n)$ and $L(t,x;\xi)$ is a polynomial in $\xi\in R^n$ of order 2μ with coefficients in $C^\infty(R_t\times R_x^n)$. In particular we have been interested in operators which are called to be of Fokker-Plank type. These were transformed by a change of independent variable into one having properties (O), (I), (II) and (III) stated in Proposition 1 and Remark of [2] (see also Theorem 3 in §2), and we could show that if an operator possesses these properties, it has a very regular right-parametrix (see Theorem 3 in §2) and hence its transpose is hypoelliptic. Applying this theorem with $I=[-1,1]$ and $\Delta=\{(t,s);-1\leqslant s<t\leqslant 1\}$, we can prove, for example, the following

Theorem 1. *Let, for real* r, $\langle r\rangle$ *be an integer such that* $r\leq\langle r\rangle<r+1$ *and* $M_j(t,x;\xi)$ *a polynomial in* $\xi\in R^n$ *of homogeneous order* j *with coefficients in* $C^\infty(R_t\times R_x^n)$. *Then both the operator*

(2) $$P=\frac{\partial}{\partial t}+\sum_{j=0}^{2\mu}t^{\langle jl/2\mu\rangle}M_j(t,x;D_x),\qquad l=0,1,\cdots,$$

and its transpose ${}^tP$ *are hypoelliptic in* $R^{n+1}=R_t\times R_x^n$, *if* l *is even and if for every compact set* K *of* R^{n+1} *there exists a constant* $\delta>0$ *such that*

(3) $$\operatorname{Re}M_{2\mu}(t,x;\xi)\geq\delta|\xi|^{2\mu},\qquad (t,x)\in K,\ \xi\in R^n.$$

For the proof we use (9) with $t\in[-1,1]$ and (t,s), $-1\leq s<t\leq 1$, and Lemmas 1 and 2 in §4.

On the other hand Kannai proved recently in [1] that the operator

$$\frac{\partial}{\partial x}-xD_y^2,\qquad D_y=-i\frac{\partial}{\partial y}$$

is hypoelliptic in the plane and moreover its transpose

$$-\frac{\partial}{\partial x}-xD_y^2$$

is not locally solvable there, of course not hypoelliptic. As an extension of this result we can give

Theorem 2. *The transpose of operator* (2), ${}^tP$, *with odd* l *is*