105. A Theorem Equivalent to the Brouwer Fixed Point Theorem

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§0. Introduction.

In this note we shall give a theorem which is equivalent to the Brouwer fixed point theorem. Such a theorem, we shall call here Theorem A, can be applied to the foundation of analysis concerning several independent variables ([1] Lemma F).

Notations used here are the same as those in [1]. Let K be the n-dimensional closed unit ball, and K_δ be the closed ball of radius δ with center $\mathbf{0}$. Further let $(S)_{-\delta}$ be the maximal closed set whose δ-neighborhood is contained in the set S. The symbol $\|\cdot\|$ denotes the ordinary euclidean norm. The Brouwer fixed point theorem for a continuous mapping on K into itself is referred to as Theorem B.

Theorem A. *Let $f(x)$ be a continuous mapping defined on K into R^n of the form*

$$f(x)=Lx+N(x),$$

where L is non-degenerated affine mapping and $\|N(x)\|\leqq\delta$.
Then $$f(K)\supset(LK)_{-\delta}.$$

For sufficiently small δ the set $(LK)_{-\delta}$ is not empty, and therefore such a continuous $f(x)$ in Theorem A may be considered as having the dimension-preserving property in some sense. Translating variables, C^1-mapping with non-vanishing Jacobian belongs to this class in local and Theorem A furnishes a lower bound of the extent of range $f(Q)$ for a small vicinity Q.

Theorem A increases in generality by certain modifications, however, we shall be interested in the fact that Theorem A which may be seen intuitively is equivalent to the Brouwer fixed point theorem.

§1. Theorem B implies Theorem A.

Proof. Let y be arbitrarily chosen from $(LK)_{-\delta}$ and fixed. Consider the mapping $x\to L^{-1}(y-N(x))$. Since $y-N(x)$ belongs to LK, this mapping is continuous on K into itself. Therefore by Theorem B there exists a fixed point $x(\in K)$ such that $L^{-1}(y-N(x))=x$ i.e. $y=Lx+N(x)$.
q.e.d.

§2. Theorem A implies TheoremB.

Proof. Suppose there exists a continuous mapping $f(x)$ on K into itself with no fixed point. Then there exists a continuous mapping