102. On An Ergodic Abelian $\mathcal{M}$-Group*)

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Let $\mathcal{M}$ be an abelian von Neumann algebra, F an $\mathcal{M}$-group (i.e. a group of automorphisms of $\mathcal{M}$). Let $[F]$ denote the full group generated by F. Choda proved in [1] that F is maximal abelian in $[F]$ if F is ergodic, abelian and free, by techniques of cross product algebras. In this note we prove, by completely different techniques, the following theorem.

Theorem. *Suppose that $\mathcal{M}$ is an abelian von Neumann algebra, and F is an ergodic abelian $\mathcal{M}$-group.*
Then:

(i) F is free.

(ii) F is maximal abelian in $[F]$.

(iii) $F' \cap [F] = F$.

(iv) $\beta \in F' \Rightarrow E(\beta, \alpha) \neq 0$ for at most one $\alpha \in F$, where $E(\beta, \alpha)$ is by definition $\sup\{F$ projection in $\mathcal{M}$: $\beta(M) = \alpha(M)$ for all $M \in \mathcal{M}$ with $FM = M\}$.

Before we prove the preceding theorem, we shall prove an auxiliary result.

Lemma 1. *Suppose that $\mathcal{M}$ is an abelian von Neumann algebra, and F is an ergodic abelian $\mathcal{M}$-group. Suppose that β is in F'. Then if α_1 and α_2 are in F with $E(\beta, \alpha_1) \neq 0$, and $E(\beta, \alpha_2) \neq 0$, we have:*

$$E(\beta, \alpha_1) = E(\beta, \alpha_2).$$

Proof. Let β agree with α_i on a non-zero projection P_i of $\mathcal{M}(i = 1, 2)$. Since F is ergodic there exists $\alpha \in F$ such that $Q = \alpha(P_1)P_2 \neq 0$. Now if $M \in \mathcal{M}$ with $\alpha(M)Q = \alpha(M)$ then $\beta(M) = \alpha_1(M)$. So for $M \in \mathcal{M}$ with $MQ = M$ we have first $\beta(M) = \alpha_2(M)$, and secondly $\beta(M) = (\alpha\beta) \times (\alpha^{-1}(M)) = \alpha\alpha_1(\alpha^{-1}(M)) = \alpha_1(M)$, where we have used both that $\beta \in F'$ and that F is abelian. Thus we see that α_1 and α_2 agree on $\alpha(P_1)P_2$. That is, any non-zero projection (of $\mathcal{M}$) on which β agrees with α_2 majorizes a non-zero projection (of $\mathcal{M}$) on which α_1 agrees with α_2. Therefore $E(\beta, \alpha_2)[I - E(\alpha_1, \alpha_2)] = 0$, or $E(\beta, \alpha_2) \leq E(\alpha_1, \alpha_2)$. By the definition of $E(\alpha_1, \alpha_2)$ we obtain

$$E(\beta, \alpha_2) \leq E(\beta, \alpha_1).$$

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