98. *On Distributive Sublattices of a Lattice*[*)]

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In his note [1], B. Jónsson gave a necessary and sufficient condition that the sublattice generated by a subset H of a modular lattice should be distributive. This condition can be proved to be equivalent to the statement that $(a\cap c)\cup(b\cap c)=(a\cup b)\cap c$ for any a, b, c in H_1, where H_1 consists of all elements which can be written as a finite join or a finite meet of elements in H. The main purpose of this paper is to prove that in order for the sublattice generated by a subset H of a lattice to be distributive it is a necessary and sufficient condition that $(a\cap c)\cup(b\cap c)=(a\cup b)\cap c$ for any a, b, c in H_2, where H_2 consists of all elements which can be written as a finite join or a finite meet of elements in H_1.

Let $\langle H\rangle$ be a sublattice generated by a nonempty subset H of a lattice L. The finite join $\bigcup_{i=1}^m x_i$ of elements $x_1, x_2, \cdots, x_m$ in H is called a $\cup$-element in $\langle H\rangle$. The set of all $\cup$-elements in $\langle H\rangle$ is denoted by $H_\cup$ and dually the set of all $\cap$-elements in $\langle H\rangle$ by $H_\cap$. One of $\cup$- or $\cap$-elements in $\langle H\rangle$ is said to be a 1st-element in $\langle H\rangle$, and the set of all 1st-elements in $\langle H\rangle$ is denoted by H_1. The finite join $\bigcup_{i=1}^m x_i$ of $\cap$-elements $x_1, x_2, \cdots, x_m$ in $\langle H\rangle$ is called a $\cup\cap$-element in $\langle H\rangle$. The set of all $\cup\cap$-elements in $\langle H\rangle$ is denoted by $H_{\cup\cap}$ and dually the set of all $\cap\cup$-elements in $\langle H\rangle$ by $H_{\cap\cup}$. One of $\cup\cap$- or $\cap\cup$-elements in $\langle H\rangle$ is said to be a 2nd-element in $\langle H\rangle$, and the set of all 2nd-elements in $\langle H\rangle$ is denoted by H_2.

Two modular laws will be denoted by

μ: $(a\cap c)\cup(b\cap c)=((a\cap c)\cup b)\cap c$, and

μ^*: $(a\cup c)\cap(b\cup c)=((a\cup c)\cap b)\cup c$.

Four distributive laws will be denoted by

δ: $(a\cap c)\cup(b\cap c)=(a\cup b)\cap c$,

δ^*: $(a\cup c)\cap(b\cup c)=(a\cap b)\cup c$,

Δ: $\bigcup_{i=1}^m(x_i\cap y)=(\bigcup_{i=1}^m x_i)\cap y$, and

Δ^*: $\bigcap_{i=1}^m(x_i\cup y)=(\bigcap_{i=1}^m x_i)\cup y$.

Theorem 1. *Let $\langle H\rangle$ be the sublattice generated by a nonempty subset H of a lattice L. In order that $\langle H\rangle$ be distributive it is necessary and sufficient that Δ holds for any $x_i\in H$ $(i=1, 2, \cdots, m)$ and any $y\in H_\cap$ (or briefly Δ holds for H), and μ and μ^* hold for any $a, b, c\in H_2$*

*) Dedicated to Professor K. Asano on his sixtieth birthday.