53. On a Theorem of I. Glicksberg

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§1. Let A be a function algebra on a compact Hausdorff space X. Some time ago Hoffman and Wermer [4] showed that the set of real parts $\mathrm{Re}\, A$ of A cannot be closed in $C_R(X)$ unless $A=C(X)$. As a consequence of the Hoffman-Wermer result, Glicksberg [3] has recently proved the following theorem: Let A be a function algebra on a compact metric space X and I be a closed ideal in A. If $A+\bar{I}$ is a closed, then $\bar{I}=I$, where $\bar{I}$ denotes the conjugate of I, i.e., $\bar{I}=\{\bar{f}; f\in I\}$. The main purpose of this paper is to give some extensions of the Glicksberg theorem in the case where X is any compact Hausdorff space.

By a function algebra on X we denote a closed subalgebra in $C(X)$ containing constant functions and separating points in X, where $C(X)$ is the Banach algebra of all complex-valued continuous functions on X with the uniform norm. Throughout this paper X will indicate a compact Hausdorff space.

Our results are following

Theorem 1. *Let A be a function algebra on a compact Hausdorff space X. Let N be a closed linear subspace in $C(X)$ and I be a closed ideal in A with $A+\bar{I}\supset N\supset I$. If $N+\bar{I}$ is closed, then $\bar{I}=I$.*

Theorem 2. *Let A be a function algebra on X. Let N be a closed linear subspace in A, I be a closed ideal in A and $N\cap I$ be an ideal in A. If $N+\bar{I}$ is closed, then $\overline{N\cap I}=N\cap I$.*

Theorem 3. *Suppose A is a function algebra on X and I, J are any two closed ideals in A. Then $I+\bar{J}$ is closed if and only if $\overline{I\cap J}=I\cap J$.*

§2. The following lemma is basic in our forthcoming proofs of these theorems.

Lemma 1. *Let A be a function algebra on X. Let N be a closed linear subspace in $C(X)$ and I be a closed ideal in A. If $N+\bar{I}$ is closed, there is $c>0$ such that $c\|g+(N\cap\bar{I})\|\le\|\mathrm{Re}\, g\|$ for any $g\in N\cap I$, where* $\mathrm{Re}\, g$ *denotes the real part of g and $\|f+(N\cap\bar{I})\|$ is the norm of the factor space $(N+\bar{I})/(N\cap\bar{I})$, i.e.,* $\|f+(N\cap\bar{I})\|=\inf_{h\in N\cap\bar{I}}\|f+h\|$.

Proof. We note first that the mapping $\Phi: f+\bar{g}+(N\cap\bar{I})\to f+(N\cap\bar{I})$ $(f\in N, g\in I)$ is well-defined as a linear mapping from the factor space $(N+\bar{I})/(N\cap\bar{I})$ to $N/(N\cap\bar{I})$. For, if $(f_1+\bar{g}_1)-(f_2+\bar{g}_2)\in N\cap\bar{I}$