66. On the Uniqueness of the Shortest Single Axiom for the Implicational Calculus of Propositions

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By the ICP (implicational calculus of propositions), we mean the implicational fragment of the classical propositional calculus. It can be axiomatized in various ways by axiom schemes and the rules of substitution and detachment as inference rules. In 1948, Jan Łukasiewicz [1] showed that the single axiom

(1) $CCCpqrCCrpCsp$

suffices to characterize the ICP, and gave a proof sketch of the fact that this is one of the shortest single axioms that can characterize the ICP. He left open, however, the question whether (1) is the unique axiom of the shortest length. In 1968, Richard Tursman [2] showed that (1) is the only 13 letter single axiom with a possible exception of

(2) $CCpqCCCqrpCsq$.

We have checked the correctness of these results using a computer, and could verify furthermore that (2) does not function as a single axiom for the ICP. Thus we came to the conclusion that (1) is the unique shortest single axiom for the ICP.

For the purpose of checking formulas, we have used models, some of which have more than one designated values. If a formula A is valid in a model but B is not so, then B can not be derived from A. Hence, in order to eliminate a candidate axiom A, we have only to show the existence of a model in which A holds but not (1). For constructing models, we used a computer. Since the implication is the only logical connective in the ICP, a model can be expressed by a matrix giving the truth table of implication, in which the designated values will be shown with the asterisk.

Tursman claimed to have eliminated using three matrices I, II and III shown below, all the 13 letter formulas from single axiom candidates except the following eleven formulas:

$CCCpCqrqCCqss$	$CCCpqpCCprCsr$
$CCCpCqrsCCsqq$	$CCCpqrCCrpCsp$
$CCpqCCCrCpspq$	$CCCpqrCsCCrpp$
$CCpqCCCrCqspq$	$CCpqCrCCCpspq$
$CCpqCCCprpCsq$	$CCpqCrCCCqspq$.
$CCpqCCCqrpCsq$	