86. *On Remainder Estimates in the Asymptotic Formula of the Distribution of Eigenvalues of Elliptic Operators*

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1. Introduction. Let Ω be a domain in R^n with boundary uniformly regular of class $m+1$. Let $\mathcal{A}=\sum a_\alpha(x)D^\alpha$ be a formally self-adjoint positively elliptic operator of order m with coefficients defined and bounded in Ω. Let A be a self-adjoint realization of $\mathcal{A}$ with domain contained in $W_2^m(\Omega)$. By $N(t)$ we denote the number of eigenvalues $\leqq t$ of A. Assuming that the highest order coefficients of A are continuous R. Beals [2] investigated the asymptotic behaviour of the resolvent kernel and spectral function of A, and as an application of his results he proved that the asymptotic formula

$$N(t)=c_0t^{n/m}+O(t^{(n-\theta)/m}), \qquad t\to\infty \tag{1.1}$$

holds for any $0<\theta<h/(h+3)$ provided that the top-order coefficients of $\mathcal{A}$ are uniformly Hölder continuous of order h. The object of this note is to improve the remainder estimate in (1.1) and prove the following theorem.

Theorem. *Suppose Ω is bounded. Let A be a self-adjoint semi-bounded realization of $\mathcal{A}$ with domain contained in $W_2^m(\Omega)$. If $m\leqq n/2$ we make the additional assumption that A satisfies the resolvent condition for $2\leqq q\leqq n/m+\varepsilon$ with some $\varepsilon>0$ ([2]), i.e. for each $\delta>0$ there are constants c_1 and c_2 such that $(A-\lambda)^{-1}$ induces a bounded operator from $L^q(\Omega)$ to $W_q^m(\Omega)$ and*

$$\|(A-\lambda)^{-1}u\|_q\leqq c_1|\lambda|^{-1}\|u\|_q$$

for all $u\in L^q(\Omega), |\lambda|\geqq c_2, |\arg\lambda|\geqq\delta$. If the highest order coefficients of $\mathcal{A}$ are uniformly continuous of order h, then

$$N(t)=c_0t^{n/m}+O(t^{(n-\theta)/m}) \tag{1.2}$$

for any $0<\theta<h/(h+2)$, where

$$c_0=(2\pi)^{-n}\int_\Omega\int_{a(x,\xi)<1}d\xi dx.$$

If the highest order coefficients of $\mathcal{A}$ belong to the class C^{1+h} in some domain containing $\bar{\Omega}$, then (1.2) holds for any $0<\theta<(h+1)/(h+3)$.

2. Outline of the proof of the main theorem.

If $m>n/2$, we have only to apply the main theorem of K. Maruo [3] to the sesquilinear form (Au, Av). Hence, in what follows we assume that $m\leqq n/2$.

Lemma 1 (R. Beals [2]). *If S and T are bounded operators in $L^2(\Omega)$ such that the ranges of S and T^* are contained in $L^\infty(\Omega)$. Then*