81. Qualitative Theory of Codimension-one Foliations

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We shall give a method of studying topological properties of integral manifolds of a completely integrable one-form.

Suppose that we are given a connected, closed $(n+1)$-manifold V^{n+1} of class C^4 with a nonsingular, completely integrable one-form ω of class $C^3, n\geqslant 1$. As in [1], a maximal connected integral manifold of ω will be called a leaf.

1. The critical cycles Σ. For each $p\in V$, by assumption, there is a local coordinate system $(x^1,\cdots,x^{n+1})$ of class C^3 in a neighborhood U of p such that $\omega|U=fdx^{n+1}$ for some positive-valued C^3 function f on U. Then the set $(U,f,(x^1,\cdots,x^{n+1}))$ is called an *$\mathcal{F}$-chart* (at p). Denote by Σ the set of zeros of the exterior derivative of ω, i.e., $\Sigma=\{p\in V\,|\,(d\omega)_p=0\}$.

Let $p\in\Sigma$. Let $(U,f,(x^1,\cdots,x^{n+1}))$ be an $\mathcal{F}$-chart at p and put

$$j_x^2(f)=\left(f_{ij}(x)\,;\begin{matrix} i\downarrow 1,\cdots,n\\ j\rightarrow 1,\cdots,n\end{matrix}\right),$$

$$j_x^3(f)=\left(f_{ij}(x),\frac{\partial}{\partial x^i}\det j_x^2(f)\,;\begin{matrix} i\downarrow 1,\cdots,n+1\\ j\rightarrow 1,\cdots,n\end{matrix}\right),$$

where $f_{ij}(x)=\partial^2 f(x)/\partial x^i\partial x^j$. Let $i=0,1,\cdots,n$. The point p is said to be of *type* (i) if the matrix $j_p^2(f)$ is nonsingular and if the number of negative eigenvalues of $j_p^2(x)$ is equal to i. We say that p is of *type* $(*)$ if $\det j_p^2(f)=0$. Of course, the type of a point of Σ is well defined independently of the choice of $\mathcal{F}$-charts. For $\lambda=0,1,\cdots,n$ or $*$, let Σ_λ denote the set of points of type (λ). Then we have $\Sigma=\Sigma_*\cup\Sigma_0\cup\cdots\cup\Sigma_n$ (disjoint union).

We shall assume that ω satisfies the following condition:

(T) *For any $p\in\Sigma_*$, there is an $\mathcal{F}$-chart $(U,f,(x^1,\cdots,x^{n+1}))$ at p such that the matrix $j_p^3(f)$ is nonsingular.*

One sees then that the same condition holds for any $\mathcal{F}$-chart at $p\in\Sigma_*$. One will also see that this condition (T) is "generic".

2. The main theorems. Assume that ω satisfies the condition (T). Then we have the following three theorems.

Theorem I. *If $\Sigma_0\neq\emptyset$ and $\Sigma_1=\emptyset$, then there exists a C^3 fibre bundle B^{n+1} over S^1 and a C^3 diffeomorphism $h:B^{n+1}\rightarrow V^{n+1}$ such that*

(i) *the fibre of B^{n+1} is a connected, simply connected, closed n-manifold of class C^3.*