133. *Note on Singular Perturbation of Linear Operators*

By Atsushi YOSHIKAWA*)
Department of Mathematics, Hokkaido University

(Comm. by Kôsaku YOSIDA, M. J. A., Oct. 12, 1972)

0. Introduction. Consider the following problem in a Banach space X:

$$\begin{cases} \partial u(t,\varepsilon)/\partial t + A(\varepsilon)u(t,\varepsilon)=0, & t>0, \\ u(0,\varepsilon)=a. \end{cases} \tag{0.1}$$

Here ε is a positive parameter, $0<\varepsilon\leqq 1$, $A(\varepsilon)=\varepsilon A+B$, and $a\in X$. We assume that A and B are closed linear operators in X with $\mathbf{D}(A)\subset\mathbf{D}(B)$ and that $-A(\varepsilon)$ with $\mathbf{D}(A(\varepsilon))=\mathbf{D}(A)$ generates a strongly continuous semi-group of bounded operators in X (i.e., of class (C_0)), uniformly with respect to ε; that is, with a constant $M>0$,

$$\|\exp(-tA(\varepsilon))\|\leqq M \tag{0.2}$$

for all $t\geqq 0$ and $0<\varepsilon\leqq 1$.

The (mild) solution of (0.1) is given by

$$u(t,\varepsilon)=\exp(-tA(\varepsilon))a, \qquad t\geqq 0,\ a\in X. \tag{0.3}$$

The map $]0,1]\in\varepsilon\mapsto u(t,\varepsilon)\in X$ is strongly continuous as seen immediately from the Trotter-Kato theorem (see Yosida [3], Kato [2]). However, $u(t,\varepsilon)$ may not be convergent as $\varepsilon\to 0$.

In the present note, we discuss a sufficient condition for the convergence of $u(t,\varepsilon)$ as $\varepsilon\to 0$. For that purpose, we introduce the set $C(p,\theta)$, $p>1$, $\theta<p-1$. $C(p,\theta)$ consists of all such elements b in $\mathbf{D}(A)$ that

$$\int_0^1 \varepsilon^\theta \sup_{t\geqq 0}\|A\exp(-tA(\varepsilon))b\|^q\, d\varepsilon<\infty. \tag{0.4}$$

It is easy to see that $C(p,\theta')\subset C(p,\theta)$ if $\theta'\leqq\theta$ and $C(q,\theta')\subset C(p,\theta)$ if $q\geqq p$, $p\theta'\leqq q\theta$.

Then we obtain the following

Theorem. *Let $b\in C(p,\theta)$ for some $p,\theta, 1<p<\infty$, $\theta<p-1$. Then* $\exp(-tA(\varepsilon))b$ *converges strongly to an element $b(t)\in X$ as $\varepsilon\to 0$, uniformly with respect to t in every compact interval. Furthermore,*

$$\exp(-tA(\varepsilon))b=b(t)+\boldsymbol{O}(\varepsilon^\rho), \qquad 0<\rho<1-\theta/(p-1). \tag{0.5}$$

Here $\boldsymbol{O}(\varepsilon^\rho)$ denotes the element in X such that $\varepsilon^{-\rho}\boldsymbol{O}(\varepsilon^\rho)$ remains bounded as $\varepsilon\to 0$, uniformly in t in every compact interval.

Let $\boldsymbol{D}=\bigcup_{p>1}\bigcup_{\theta<p-1}C(p,\theta)$. Then we immediately have

Corollary. *Let $\boldsymbol{D}$ be dense in X. Then there is an extension B_1*

---

*) Supported by the Sakkokai Foundation.