147. Large Subfields and Small Subfields

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Let L/K be any extension of fields. As in the module theory (Lambek [2], p. 93), a subfield E of L/K (i.e. a subfield of L containing the field K) is called *a small subfield*, if, for any subset A of L, $E(A)=L$ implies $K(A)=L$, where $K(A)$ is the subfield generated by A over K. Further, a subfield F is called *a large subfield* of L/K, if, for any subfield H of L, $F\cap H=K$ implies $H=K$. We shall discuss the existence of a minimal large subfield and a maximal small subfield. The method used here is similar to the one in the group theory (Kurosh [1], p. 217).

A field H is called *a proper subfield* of L/K, if $K\subsetneqq H\subsetneqq L$.

Theorem 1. *Assume that any proper subfield of L/K contains a proper minimal subfield. Then, the subfield F, generated by all the proper minimal subfields of L/K, is a unique minimal large subfield of L/K.*

Proof. We shall show that, for any subfield G of L/K, G is large if and only if $G\supseteqq F$.

If $G\not\supseteqq F$, there exists a minimal subfield M such that $G\not\supseteqq M$. Then, by the minimality of M, $G\cap M=K$. Since $K\subsetneqq M$, G is not large.

On the other hand, let G be not large. Then, there exists a subfield $H\supsetneqq K$ of L/K, such that $G\cap H=K$. By assumption, there exists a minimal subfield M such that $H\supseteqq M\supsetneqq K$. Then, $M\cap G\subseteqq H\cap G=K$. This shows that $G\not\supseteqq M$, and so $G\not\supseteqq F$. Q.E.D.

If L/K is an algebraic extension, the assumption of Theorem is always satisfied. Further, if L/K is a Galois extension, the above field F is the Frattini subfield defined by Neukirch ([3], p. 41). On the other hand, by Lüroth's Theorem, any proper subfield of a rational function field $K(X)/K$ is also a rational function field of the type $K(Y)$ ($Y\in K(X)$: transcendental over K). Therefore, a rational function field $L=K(X)$ does not satisfy the assumption of Theorem 1.

As a dual of Theorem 1, we have

Theorem 2. *Assume that any proper subfield of L/K is contained in a proper maximal subfield of L/K. Then, the intersection E of all the proper maximal subfields of L/K is a unique maximal small subfield of L/K.*

Since the proof is also dual to that of Theorem 1, we do not repeat it.