

142. On the Automorphism Group of the Full Transformation Semigroups

By Nobuko IWAHORI and Hirosi NAGAO

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Let $\Sigma(\Omega)$ be the semigroup of all the mapping of a set Ω into Ω and $\mathfrak{S}(\Omega)$ the symmetric group on Ω . For $a \in \Omega$ we denote by ε_a the element of $\Sigma(\Omega)$ such that $\varepsilon_a(x) = a$ for every $x \in \Omega$. Let $E = \{\varepsilon_a \mid a \in \Omega\}$.

The purpose of this note is to prove the following:

Theorem. *Let S be a subsemigroup of $\Sigma(\Omega)$ containing E and let θ be an isomorphism of S into $\Sigma(\Omega)$. If $\theta(S)$ contains E , then θ can be extended to an inner automorphism of $\Sigma(\Omega)$, i.e., there is an element σ of $\mathfrak{S}(\Omega)$ such that $\theta(\gamma) = \sigma\gamma\sigma^{-1}$ for all $\gamma \in S$. In case Ω is finite, the weaker assumption that $\theta(S) \cap E \neq \emptyset$ implies the same conclusion.*

Denote by $\Sigma^*(\Omega)$ the subsemigroup $\Sigma(\Omega) - \mathfrak{S}(\Omega)$ of $\Sigma(\Omega)$. From the theorem we obtain, immediately the following:

Corollary¹⁾. *Every automorphism of $\Sigma(\Omega)$ or of $\Sigma^*(\Omega)$ is inner. Thus the automorphism groups of $\Sigma(\Omega)$ and $\Sigma^*(\Omega)$ are both isomorphic to $\mathfrak{S}(\Omega)$.*

Before proving the theorem, we remark that an element ε_a of E satisfies the following equations for all $\gamma \in \Sigma(\Omega)$:

$$(1) \quad \varepsilon_a \gamma = \varepsilon_a$$

$$(2) \quad \gamma \varepsilon_a = \varepsilon_{\gamma(a)}$$

It follows from (2) that if an element δ of $\Sigma(\Omega)$ satisfies $\delta \varepsilon_a = \delta$ for some $a \in \Omega$, then $\delta = \varepsilon_{\delta(a)}$ is an element of E .

Proof of theorem. Let θ be an isomorphism of S into $\Sigma(\Omega)$ such that $\theta(S) \cap E \neq \emptyset$. Then there is an element $\beta \in S$ such that $\theta(\beta) = \varepsilon_b$ for some $b \in \Omega$. Since $\varepsilon_a \beta = \varepsilon_a$ for every $a \in \Omega$, $\theta(\varepsilon_a) \varepsilon_b = \theta(\varepsilon_a)$. By the above remark $\theta(\varepsilon_a) \in E$ and hence $\theta(E) \subset E$. If Ω is finite then this implies $\theta(E) = E$. In the case Ω is infinite, if we assume that $\theta(S) \supset E$, then in the same way as above, we get $\theta^{-1}(E) \subset E$ and hence $\theta(E) = E$. Let $\theta(\varepsilon_a) = \varepsilon_{\sigma(a)}$. Then, since $\theta(E) = E$, it follows that σ is in $S(\Omega)$. Now from (2) we have

$$\begin{aligned} \theta(\gamma) \varepsilon_{\sigma(a)} &= \varepsilon_{\sigma(\gamma(a))} \\ &= \varepsilon_{\theta(\gamma)\sigma(a)} \end{aligned}$$

for any $\gamma \in S$. Thus $\theta(\gamma)\sigma = \sigma\gamma$ and hence $\theta(\gamma) = \sigma\gamma\sigma^{-1}$.

Remark. The following two examples show that the assumptions in the theorem are not superfluous.

1) This result was first obtained by the first author by making use of the result in [1].