25. On Some Examples of Non-normal Operators. II

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1. Introduction. Consider a (bounded linear) operator T acting on a Hilbert space $\mathfrak{H}$. As usual, cf. [3], we shall call

$$W(T)=\{(Tx \mid x)\,;\, \|x\|=1, x \in \mathfrak{H}\}$$

the *numerical range* of T. An operator T is called a *convexoid* if $\overline{W}(T) = \mathrm{co}\,\sigma(T)$, where $\overline{W}(T)$ is the closure of $W(T)$, $\sigma(T)$ is the spectrum of T and $\mathrm{co}\,M$ is the convex hull of a set M in the complex plane. We shall also say that T satisfies the *condition* (G_1) (in symbol, $T \in (G_1)$) if

$$\|(T-\lambda)^{-1}\| \leqq \frac{1}{\mathrm{dist}\,(\lambda, \sigma(T))} \tag{1}$$

for any $\lambda \notin \sigma(T)$. If $T \in (G_1)$, then T is a convexoid, cf. [1] and [7].

In a recnt paper [4], Luecke introduced a new class of operators: $T \in \mathfrak{R}$ if

$$\|(T-\lambda)^{-1}\| = \frac{1}{\mathrm{dist}\,(\lambda, W(T))} \tag{2}$$

for any $\lambda \notin \overline{W}(T)$. He proved the following theorem:

Theorem A (Luecke). *$T \in \mathfrak{R}$ if and only if $\partial W(T) \subset \sigma(T)$, where ∂M is the boundary of M.*

Luecke's definition and theorem are interested in their own right; they establish a closed connection between a growth condition of resolvents and a spectral property of operators. However, in the light of the theory of seminormal operators, Luecke's class $\mathfrak{R}$ is rather restrictive. Even in the case of finite dimensional spaces, $\mathfrak{R}$ consists of the multiples of the identity, so that general normal operators are excluded by $\mathfrak{R}$.

In the present note, we shall introduce a class of operators which is defined by a growth condition and includes both (G_1) and $\mathfrak{R}$. For this purpose, we need to define the *hen-spectrum* $\tilde{\sigma}(T)$ of an operator T by $\tilde{\sigma}(T) = ([\sigma(T)^c]_\infty)^c$ where M^c is the complement of M and $[M]_\infty$ the component of the infinity (unbounded component) of M. Clearly, $[M]_\infty$ is unique if M is bounded. By the definition, it is clear that $\tilde{\sigma}(T)$ is a compact set in the plane and contains $\sigma(T)$. Furthermore, we need the following idea due to Saito [6]: T is called an operator satisfying the *condition* (G_1) *for* M if

$$\|(T-\lambda)^{-1}\| \leqq \frac{1}{\mathrm{dist}\,(\lambda, M)} \tag{3}$$