41. *A Remark on a Sufficient Condition for Hypoellipticity*

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1. Introduction. Let $P=P(x, D_x)=\sum_{|\alpha|\le m} a_\alpha(x)D_x^\alpha$ be a differential operator where $x=(x_1, \cdots, x_n)$ is a point of a open subset Ω in real n-space R_x^n, $\alpha=(\alpha_1, \cdots, \alpha_n)$ is a multi-index with its length $|\alpha|=\alpha_1+\cdots+\alpha_n$ and $D_x^\alpha=(-i\partial/\partial x_1)^{\alpha_1}\cdots(-i\partial/\partial x_n)^{\alpha_n}$. For $\xi\in R^n$ we denote $\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$, $|\xi|=(\xi_1^2+\cdots+\xi_n^2)^{1/2}$, $\langle\xi\rangle=1+|\xi|$, $P(x,\xi)=\sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ and $P_{(\beta)}^{(\alpha)}(x,\xi)=D_\xi^\alpha(iD_x)^\beta P(x,\xi)$.

Simple and weak sufficient conditions for hypoellipticity are given by L. Hörmander which include not only differential operators but also pseudo-differential operators ([2] § 4 Theorem 4.2, p. 164). In this note we shall give a slightly different sufficient condition for hypoellipticity which is stated by using a basic weight function depending also on the x-variable instead of $\langle\xi\rangle$ only. The usage such a basic weight function is effective for study of asymptotic behavior of spectral function of hypoelliptic differential operator which will appear in a forthcoming paper.

We confine ourselves in case of differential operators but it seems quite possible to extend it in case of pseudo-differential operators, because the proof of the main theorem depends on a construction of a parametrix just along the arguments in [1] and [2]. I wish to thank Mr. M. Nagase for his advice through discussion.

2. Theorem and outline of the proof. Theorem. *Let $P(x,\xi)$ be written in the sum $P(x,\xi)=p_0(x,\xi)+p_1(x,\xi)$ where $p_0=p_0(x,\xi)$ and $p_1=p_1(x,\xi)$ satisfy the following conditions:*

(2.1) *The coefficients are in C^∞.*

For any $x\in\Omega$ and α and β there exist the constants $C_{x,\alpha,\beta}>0$, $C_x>0$, and $A_x>0$ such that

(2.2) $$|p_{0(\beta)}^{(\alpha)}(x,\xi)|\leqq C_{x,\alpha,\beta}|p_0(x,\xi)|^{1-\rho|\alpha|+\delta|\beta|}$$

(2.2)′ $$|p_{1(\beta)}^{(\alpha)}(x,\xi)|\leqq C_{x,\alpha,\beta}|p_0(x,\xi)|^{1-\rho(|\alpha|+1)+\delta(|\beta|+1)} \quad \text{for } |\xi|\geqq A_x,$$

where ρ and δ are some constants depending only on $P(x,D)$ and satisfying $0\leqq\delta<\rho\leqq 1$,

(2.3) $$|p_0(x,\xi)|\geqq C_x|\xi|^{m'} \quad 0<m'\leqq m, \qquad \text{for } |\xi|\geqq A_x,$$

(2.4) $$m'\delta<1,$$

and $C_{x,\alpha,\beta}$, C_x and A_x are bounded when x is in compact subset of Ω. Then the operator $P(x,D_x)$ is hypoelliptic: $u\in\mathcal{D}'(\Omega)$ satisfying the equa-