40. On G_δ-Sets in the Product of a Metric Space and a Compact Space. II

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In [2] we defined G_δ-*space* as a topological space which is homeomorphic to a G_δ-set in the product of a metric space and a compact Hausdorff space and proved that an M-space is a G_δ-space if and only if it is a p-space. We also left it as an open problem to give an internal characterization to G_δ-spaces. The purpose of this note is to give such characterizations. All spaces in this note are at least Hausdorff, and all maps are continuous. As for general terminologies and symbols in general topology, see [1].

Definition 1. Let $\{\mathcal{W}_i \mid i=1, 2, \cdots\}$ be a sequence of open covers of a space X and $\mathcal{F}$ a filter in X. Then $\mathcal{F}$ is said to be *Cauchy w.r.t.* $\{\mathcal{W}_i\}$ provided for each i there is $F \in \mathcal{F}$ and $W \in \mathcal{W}_i$ with $F \subset W$. Suppose S is a closed set of X. If every maximal closed filter $\mathcal{F}$ in X which is Cauchy w.r.t. $\{\mathcal{W}_i\}$ and contains S as an element converges, then S is said to be *complete w.r.t.* $\{\mathcal{W}_i\}$. We may drop the word 'w.r.t. $\{\mathcal{W}_i\}$' discussing Cauchy filter or complete closed set if there is no fear of confusion.

Definition 2. Let X be a space with a sequence $\{\mathcal{W}_i\}$ of open covers and f a map from X onto a space Y. If for every $y \in Y$ and for every maximal closed filter $\mathcal{F}$ in X, Cauchy w.r.t. $\{\mathcal{W}_i\}$ satisfying $f^{-1}(y) \notin \mathcal{F}$, there is $G \in \mathcal{F}$ such that $y \notin \overline{f(G)}$, then f is said to be *closed w.r.t.* $\{\mathcal{W}_i\}$. Obviously each closed map from X onto Y is closed w.r.t. every sequence $\{W_i\}$ of open covers of X.

Theorem 1. *A Tychonoff space X is a G_δ-space (namely homeomorphic to a G_δ-set in the product of a metric space and a compact Hausdorff space) if and only if there is a sequence $\{\mathcal{W}_i \mid i=1, 2, \cdots\}$ of open covers of X and a map f from X onto a metric space M such that*
(i) *for each $y \in M$, $f^{-1}(y)$ is complete w.r.t. $\{\mathcal{W}_i\}$,*
(ii) *f is closed w.r.t. $\{\mathcal{W}_i\}$.*

Proof. *Necessity.* Let X be a G_δ-set in the product space $C \times M$ of a compact Hausdorff space C and a metric space M. Suppose $X = \bigcap_{i=1}^{\infty} U_i$, where U_i is an open set of $C \times M$. We denote by π_1 and π_2 the projections from $C \times M$ onto C and M respectively. For each point

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