39. On G_δ-Sets in the Product of a Metric Space and a Compact Space. I

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We have proved in [8] that a topological space is paracompact (Hausdorff) and M if and only if it is homeomorphic to a closed set of the product of a metric space and a compact Hausdorff space. A similar characterization for general M-spaces may be obtained, but it is still an open question whether 'M-space' is characterized as a closed set in the product of a metric space and a countably compact space (see [9]). In this brief note we are going to turn our attention to G_δ-sets in the product of a metric space and a compact space. Although we are not successful yet in finding an internal characterization of those sets, they seem deeply related with A. V. Arhangelskii's p-spaces (see [1]) as will be seen in the following discussion. All spaces in this paper are at least Hausdorff, and all maps (=mappings) are continuous. As for the concept of M-space (due to K. Morita) the reader is referred to [4]. For general terminologies and symbols in general topology (see [6]).

Theorem 1. *An M-space X is homemorphic to a G_δ-set in the product of a metric space and a compact Hausdorff space if and only if it is a p-space.*

Proof. It is known that the product of a metric space and a compact Hausdorff space is paracompact and p, and it is also easy to see that every G_δ-set of a p-space is p. Therefore we shall prove only the 'if' part of the theorem. Assume that X is M and p at the same time. Then by Morita's theorem [4] there is a quasi-perfect map f from X onto a metric space Y. (Namely f is closed and continuous, and $f^{-1}(y)$ is countably compact for each $y \in Y$.) By D. Burke's theorem [3] there is a sequence $\mathcal{V}_1, \mathcal{V}_2, \cdots$ of open covers of X such that

(i) if $x \in V_i \in \mathcal{V}_i$, $i=1,2,\cdots$, then $K=\bigcap_{i=1}^\infty \overline{V}_i$ is compact,

(ii) for every open set U containing K, there is k for which $\bigcap_{i=1}^k \overline{V}_i \subset U$.

We may assume without loss of generality that each $\mathcal{V}_i$ consists of cozero open sets (=complements of zero sets of real-valued continuous functions defined on X), because X is a Tychonoff space (which is implied by the fact that X is p).

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