34. Continuity of the Map $S \to |S|$ for Linear Operators*)

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This note is concerned with the continuity of the map $|\cdot|$ from $B(H,H')$ to $B_{sa}(H)$ given by $|S|=(S^*S)^{1/2}$; here $B(H,H')$ denotes the set of all bounded linear operators on a Hilbert space H to another Hilbert space H', and $B_{sa}(H)$ the set of all bounded selfadjoint operators in H. We shall prove the following results.

I. *The map $|\cdot|$ is almost Lipschitz-continuous,in the sense that*

$$\||S|-|T|\| \leq \frac{2}{\pi}\|S-T\|\left(2+\log \frac{\|S\|+\|T\|}{\|S-T\|}\right),$$

where $\|\cdot\|$ denotes the operator norm.

II. *If both H and H' are infinite-dimensional, the map $|\cdot|$ is not Lipschitz-continuous in the operator norm, even when $H'=H$ and $|\cdot|$ is restricted on $B_{sa}(H)$.*

III. *For each integer $n \geq 1$, there is a holomorphic family of operators $S(t) \in B_{sa}(H)$, $-1<t<1$, where H is a finite-dimensional Hilbert space, with the following properties.* (i) $0<|S(t)|<2I$, (ii) $\|dS(t)/dt\|<1$, *and* (iii) $\|[d|S(t)|/dt]_{t=0}\|>n^2$. *Note that $|S(\cdot)|$ is also holomorphic.*

IV. *There exists a family $T(t)$, $-1<t<1$, of selfadjoint operators in a separable Hilbert space H such that $T(t)^{-1}$ exists as a bounded operator, $T(t)^{-1}$ is norm-continuously differentiable in $t \in (-1,1)$, but $|T(t)^{-1}|$ is not weakly differentiable at $t=0$.*

Remarks. 1. Propositions I and II answer some questions that appear to have been open, see e.g. Reed and Simon [1, p. 197].

2. In II it suffices to consider the special case mentioned at the end. The result for this special case is, however, a direct consequence of III.

3. IV answers a question raised by Cooper [2].

4. It seems difficult to construct a twice differentiable family $T(t)^{-1}$ with properties similar to those stated in IV. The reason is that $\|A\|$ used in (8) below grows very fast with n. Thus it is not known to the author whether or not the continuous differentiability of $T(t)^{-1}$ can be replaced by a higher order differentiability or even by analyticity.

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