66. *An Ergodic Theorem for a Semigroup of Linear Contractions*

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1. The purpose of the present paper is to extend a general ergodic theorem [1] and a general ergodic theorem of Abel type [4] in the discrete case to one in the continuous case.

2. Consider a σ-finite measure space $(X, \mathcal{F}, \mu)$ and also a measure space $(R^+, \mathcal{M}, dt)$ where $R^+=[0, \infty)$, $\mathcal{M}$ is the σ-algebra of all Lebesgue measurable subsets of R^+ and dt the Lebesgue measure on $\mathcal{M}$. Let L_1 be the real or complex Banach space of all equivalence classes of real or complex valued integrable functions on X.

Let $\{T_t : t \in R^+\}$ be a strongly continuous semigroup of linear contractions on L_1. Then it is known that, given $f \in L_1$, there exists a $\mathcal{M}\otimes\mathcal{F}$-measurable function g on $R^+\otimes X$ such that, for every t, $g(t, x) =(T_t f)(x)$ for a.a.x. Such a function g is uniquely determined up to a set of $dt\otimes d\mu$-measure zero. In what follows, $g(t, x)$ will be denoted by $(T_t f)(x)$. Then, by Fubini's theorem it is shown that, for a.a.x chosen suitably, $(T_t f)(x)$ is Lebesgue integrable on any bounded subinterval of R^+.

A family $\{p_t : t \in R^+\}$ of nonnegative measurable (not necessarily integrable) functions on X is called $\{T_t\}$-*admissible* if it satisfies

(i) *Admissibility*. $f \in L_1$ and $|f| \leqslant p_t$ for some t imply $|T_t f| \leqslant p_{s+t}$ for all s;

(ii) *Continuity*. There exists a strictly positive L_1-function p such that $\lim_{t\to s} \| |p_t - p_s| \wedge p \| = 0$ for all s, where $q \wedge p$ means $\min(q, p)$.

Lemma 1. *Let* $\{p_t : t \in R^+\}$ *be* $\{T_t\}$*-admissible. Then there exists an* $\mathcal{M}\otimes\mathcal{F}$*-measurable function* g *on* $R^+\otimes X$ *such that, for every* t, $g(t, x) = p_t(x)$ *for* a.a.x. *Such a function* g *is uniquely determined up to a set of* $dt\otimes d\mu$*-measure zero.*

Proof. Define $p_{t,n}(x) = p_{[nt]/n}(x)$, where $[nt]$ is the integral part of nt. Then $p_{t,n}(x)$ is $\mathcal{M}\otimes\mathcal{F}$-measurable and, for every t,

$$\lim_{n\to\infty} \| |p_{t,n} - p_t| \wedge p \| = 0.$$

On the other hand, since

$$|p_{t,m} - p_{t,n}| \wedge p \leqslant 2(|p_{t,m} - p_t| \wedge (p/2)) + 2(|p_{t,n} - p_t| \wedge (p/2)),$$

so