154. Estimates from $W_{p,\alpha}$ to $W_{q,\beta}$ for the Solutions of the Petrovskii Well Posed Cauchy Problems

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1. Introduction and results.

In this note, we shall consider the Cauchy problem

$$(1)\qquad \begin{cases} \dfrac{\partial u(t,x)}{\partial t}=P(D)u(t,x) & (t,x)\in(0,\infty)\times R^n, \\ u(0,x)=u_0(x) & x\in R^n. \end{cases}$$

Here $P(D)$ is the pseudo-differential operator of order d, that is,

$$(2)\qquad P(D)u=F^{-1}(S\hat{u}),\qquad u\in\mathcal{S}'^N,$$

where $S=(s_{ij})_{1\leqslant i,j\leqslant N}$ is the $N\times N$ matrix of functions s_{ij} in $C^\infty(R^n)$ which satisfy, for all multi-indices $\sigma=(\sigma_1,\cdots,\sigma_n)$,

$$(3)\qquad |D^\sigma s_{ij}(y)|\leqslant C_\sigma(1+|y|)^{d-|\sigma|}$$

where C_σ are constants depending on σ, $D^\sigma=(\partial/\partial y_1)^{\sigma_1}\cdots(\partial/\partial y_n)^{\sigma_n}$ and $|\sigma|=\sigma_1+\cdots+\sigma_n$. The matrix S will be called the symbol of P. In the above, $\mathcal{S}'^N$, F^{-1} and $\hat{u}$ denote the space of all N-tuples of distributions in the dual space $\mathcal{S}'$ of the Schwartz space $\mathcal{S}$, the inverse Fourier transformation and the Fourier transform of u, respectively. We assume that the order d of P is positive.

Let $\lambda_j(y)$ denote the eigenvalues of $S(y)$ for $j=1,2,\cdots,N$. We say that the Cauchy problem (1) is Petrovskii well posed if

$$(4)\qquad \operatorname{Re}\lambda_j(y)\leqslant\Lambda,\qquad 1\leqslant j\leqslant N,\ y\in R^n,$$

are valid for some constant Λ. When the Cauchy problem (1) is Petrovskii well posed, we can solve the problem in $\mathcal{S}'^N$ and the solution can be written as

$$(5)\qquad u(t)=E(t)u_0=F^{-1}(\exp(tS)\hat{u}_0)\qquad \text{for } u_0\in\mathcal{S}'^N.$$

We call the operator $E(t): u_0\to u(t)$ the solution operator.

Let $1\leqslant p\leqslant\infty$. For $u\in L_p^N$ (the space of all N-tuples of functions in $L_p(R^n)$), we set

$$\|u\|_p=\begin{cases}\left(\int_{R^n}|u(x)|^p\,dx\right)^{1/p} & \text{if } p<\infty \\ \operatorname{ess\,sup}\{|u(x)|;\ x\in R^n\} & \text{otherwise.}\end{cases}$$

For $\alpha\geqslant 0$, let $v_\alpha(y)=(1+|y|^2)^{\alpha/2}$ and

$$\|u\|_{p,\alpha}=\|F^{-1}(v_\alpha\hat{u})\|_p\qquad \text{for } u\in L_p^N.$$

We define $W_{p,\alpha}^N=\{u\in L_p^N;\ \|u\|_{p,\alpha}<\infty\}$.

Henceforth, for given p and q, we set $\gamma(p,q)=\max(1/2-1/p,\ 1/q-1/2,\ 0)$. Our results are the following.