3. *The Fundamental Solution for a Degenerate Parabolic Pseudo-Differential Operator*

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Introduction. In the present paper we shall construct the fundamental solution $U(t)$ for a degenerate parabolic pseudo-differential equation of the form

$$(0.1)\qquad \begin{cases} Lu=\dfrac{\partial u}{\partial t}+p(t\,;x,D)u=0 & \text{in } (0,T)\times R^n \\ u|_{t=0}=u_0 \end{cases}$$

where $p(t\,;x,D)$ is a pseudo-differential operator of class $\mathcal{E}_t^0(S^m_{\rho,\delta})$ which satisfies conditions (cf. [1], [5]):

(i) There exist constant C and m' $(0\leqq m'\leqq m)$ such that

$$(0.2)\qquad \operatorname{Re} p(t\,;x,\xi)\geqq C\langle\xi\rangle^{m'} \qquad \text{uniformly in } t\quad (0\leqq t\leqq T).$$

(ii) For any multi index $\alpha=(\alpha_1,\cdots,\alpha_n)$, $\beta=(\beta_1,\cdots,\beta_n)$ there exists a constant $C_{\alpha,\beta}$ such that

$$(0.3)\qquad |p^{(\alpha)}_{(\beta)}(t\,;x,\xi)/\operatorname{Re} p(t\,;x,\xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|} \qquad \text{uniformly in } t\quad (0\leqq t\leqq T),$$

where $p^{(\alpha)}_{(\beta)}(t\,;x,\xi)=(\partial/\partial\xi_1)^{\alpha_1}\cdots(\partial/\partial\xi_n)^{\alpha_n}(-i\partial/\partial x_1)^{\beta_1}\cdots(-i\partial/\partial x_n)^{\beta_n}p(t\,;x,\xi)$, $|\alpha|=|\alpha_1|+\cdots+|\alpha_n|$, $|\beta|=|\beta_1|+\cdots+|\beta_n|$ and $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$.

The fundamental solution $U(t)$ will be found as a pseudo-differential operator of class $S^0_{\rho,\delta}$ with parameter t. Then the solution of the Cauchy problem (0.1) is given by $u(t)=U(t)u_0$ for $u_0\in L^2$ and moreover for $u_0\in L^p$ $(1<p<\infty)$ in case $\rho=1$, using that operators of class $S^m_{\rho,\delta}$ are bounded in L^2 for $0\leqq\delta<\rho\leqq 1$, in L^p for $0\leqq\delta<1$, $\rho=1$ (see [1]–[3]).

The solution $U(t)$ is given in the form $U(t)=e(t,0\,;x,D)$ where $e(t,s\,;x,D)$ is the solution of an operator equation

$$\begin{cases} L_{x,t}e(t,s\,;x,D)=0 & \text{in } t>s\quad (0\leqq s<t\leqq T) \\ e(t,s\,;x,D)|_{t=s}=I, \end{cases}$$

which can be reduced to an integral equation of the form

$$(0.4)\qquad r_N(t,s\,;x,D)+\varphi(t,s\,;x,D)+\int_s^t r_N(t,\sigma\,;x,D)\varphi(\sigma,s\,;x,D)d\sigma=0,$$

where $r_N(t,s\,;x,D)$ is a known operator of class $S^{m-(\rho-\delta)(N+1)}_{\rho,\delta}$. To solve (0.4), we shall calculate the symbol for multi product of pseudo-differential operators in precise form by using oscillatory integrals in [4] and [6].

1. Notations and Theorem. We shall denote by $S^m_{\rho,\delta}$ $(0\leqq\delta<\rho\leqq 1,$