34. *Note on Products of Symmetric Spaces*

By Yoshio TANAKA

(Comm. by Kinjirô KUNUGI, M. J. A., Feb. 12, 1974)

1. Introduction: In [7, Corollary 4.4], we have shown that *if X is a locally compact, symmetric space and Y is a symmetric space, then $X \times Y$ is a symmetric space.*

In this note, we shall show this result is the best possible. Namely, we have

Theorem. *Let X be a regular space. Then the following are equivalent.*

(a): *X is a locally compact, symmetric space.*

(b): *$X \times Y$ is a symmetric space for every symmetric space Y.*

According to A. V. Arhangel'skii [1], a space X is *symmetric*, if there is a real valued, non-negative function d defined on $X \times X$ satisfying the following:

(1): $d(x,y)=0$ whenever $x=y$, (2): $d(x,y)=d(y,x)$, and (3): $A \subset X$ is closed in X whenever $d(x,A)>0$ for any $x \in X-A$.

Metric spaces and semi-metric spaces are symmetric.

We assume all spaces are Hausdorff.

2. Proof of Theorem. For proof, we use the method in [3, Theorem 2.1].

The implication (a)$\Rightarrow$(b) follows from [7, Corollary 4.4].

To prove the implication (b)$\Rightarrow$(a), suppose that $X \times Y$ is a symmetric space for every symmetric space Y, and that a regular space X is not locally compact.

Since a countably compact, symmetric space is compact [5, Corollary 2], X is not a locally countably compact space.

Then there are a point $x_0 \in X$ and a local base $\{U_\alpha : \alpha \in A\}$ at x_0 such that each $\overline{U}_\alpha$ is not countably compact. Hence, for each $\alpha \in A$, there is an infinite, discrete closed subset $\{x_i^\alpha : i=1,2,\cdots\}$ of X such that $x_i^\alpha \in \overline{U}_\alpha$.

Topologize A with discrete topology. Let $A_i = A \times \{i\}$ for each positive integer i, and let $\sum_{i=1}^\infty A_i$ be the topological sum of A_i. Let $X_1 = \sum_{i=1}^\infty A_i \cup \{\infty\}$ and let $\{V_j(\infty) : j=1,2,\cdots\}$ be a local base at the point ∞, where $V_j(\infty)=\{\infty\} \cup \bigcup_{k \geq j} A_k$. Then a regular space X_1 has a σ-locally-finite base. By J. Nagata and Yu. M. Smirnov Metrization Theorem, X_1 is a metrizable space.

Let $[0,\omega]$ be the ordinal space, where ω is the first countable ordinal number.