108. On Common Fixed Point Theorems of Mappings

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In his recent book [1], V. I. Istrătesçu proved some common fixed point theorems about contraction mappings. In this paper, we shall generalize his results.

Let (X,ρ) be a complete metric space, and T_k $(k=1,2,\cdots,n)$ a family of mappings of X into itself.

Theorem 1. *If T_k $(k=1,2,\cdots,n)$ satisfies*

1) $T_kT_l=T_lT_k$ $(k,l=1,2,\cdots,n)$,

2) *There is a system of positive integers $m_1,m_2,\cdots,m_n$ such that*

$$\begin{aligned}&\rho(T_1^{m_1}T_2^{m_2}\cdots T_n^{m_n}x, T_1^{m_1}T_2^{m_2}\cdots T_n^{m_n}y)\\ &\quad\le\alpha\rho(x,y)+\beta[\rho(x, T_1^{m_1}T_2^{m_2}\cdots T_n^{m_n}x)\\ &\qquad+\rho(y, T_1^{m_1}T_2^{m_2}\cdots T_n^{m_n}y)]+\gamma[\rho(x, T_1^{m_1}T_2^{m_2}\cdots T_n^{m_n}y)\\ &\qquad+\rho(y, T_1^{m_1}T_2^{m_2}\cdots T_n^{m_n}x)]\end{aligned}\tag{1}$$

for every x, y of X, where α,β,γ are non-negative and $\alpha+2\beta+2\gamma<1$, then T_k $(k=1,2\cdots,n)$ have a unique common fixed point.

Proof. To prove Theorem, we use I. Rus theorem [2]. Let $U=T_1^{m_1}T_2^{m_2}\cdots T_n^{m_n}$, then by (1), we have

$$\rho(Ux, Uy)\le\alpha(x,y)+\beta[\rho(x,Ux)+\rho(y,Uy)]+\gamma[\rho(x,Uy)+\rho(y,Ux)]$$

for all x, y of X. Hence by I. Rus theorem, U has a unique fixed point ξ in X. Therefore $U\xi=\xi$, then we have

$$T_i(U\xi)=T_i\xi\qquad(i=1,2,\cdots,n).\tag{2}$$

By the commutativity of $\{T_k\}$, (2) implies

$$U(T_i\xi)=T_i\xi.$$

Since U has a unique fixed point ξ, we obtain $T_i\xi=\xi$ $(i=1,2,\cdots,n)$. Hence ξ is a common fixed point of the family $\{T_k\}$.

Let ξ, η be common fixed points of $\{T_k\}$, then by (1), we have

$$\begin{aligned}\rho(\xi,\eta)=\rho(U\xi,U\eta)&\le\alpha\rho(\xi,\eta)\\ &+\beta[\rho(\xi,U\xi)+\rho(\eta,U\eta)]+\gamma[\rho(\xi,U\eta)+\rho(\eta,U\xi)],\end{aligned}$$

which implies

$$\rho(\xi,\eta)\le\alpha\rho(\xi,\eta)+2\gamma\rho(\xi,\eta).$$

From $\alpha+2\gamma<1$, we have $\rho(\xi,\eta)=0$, i.e. $\xi=\eta$. We have the uniqueness, and we complete the proof.

Theorem 2. *If $\{T_k\}$ satisfies the conditions:*

1) *$T_1T_2\cdots T_n$ commutes with every T_i,*

2) *for every x, y of X,*