139. On Characterizations of Spaces with G_δ-diagonals

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A space X is called to have a G_δ-diagonal if the diagonal Δ in $X\times X$ is a G_δ-set. A space X is called to have a regular G_δ-diagonal if Δ is a regular G_δ-set, that is, Δ is written by the following:

$$\Delta=\cap\{U_n/n\in N\}=\cap\{\overline{U}_n/n\in N\},$$

where U_n's are open sets containing Δ in $X\times X$ and N denotes the set of all natural numbers. Ceder in [1] characterized a G_δ-diagonal as follows:

Lemma 1. *A space X has a G_δ-diagonal iff (=if and only if) there is a sequence $\{\mathcal{U}_n/n\in N\}$ of open coverings of X such that for each point p in X*

$$p=\cap\{S(p,\mathcal{U}_n)/n\in N\}.$$

According to Zenor's result in [2], a regular G_δ-diagonal is characterized as follows:

Lemma 2. *A space X has a regular G_δ-diagonal iff there is a sequence $\{\mathcal{U}_n/n\in N\}$ of open coverings of X such that if p,q are distinct points in X, then there are an integer n and open sets U and V containing p and q, respectively, such that no member of $\mathcal{U}_n$ intersects both U and V.*

The object of the present paper is to characterize spaces with G_δ- or regular G_δ-diagonal by virtue of above lemmas as images of metric spaces under open mappings with some properties.

Theorem 1. *A space X has a G_δ-diagonal iff there is an open mapping (single-valued) f from a metric space T onto X such that*

$$d(f^{-1}(p),f^{-1}(q))>0 \textit{ for distinct points } p,q\in X.$$

Proof. Only if part: Define T as follows:

$$T=\{(\alpha_1,\alpha_2,\cdots)\in N(A)/\cap\{U^n_{\alpha_n}/n\in N\}\neq\phi\},$$

where $\{\mathcal{U}_n=\{U^n_\alpha/\alpha\in A\}/n\in N\}$ is a sequence of open coverings of X satisfying the condition in Lemma 1. If we define a mapping $f:T\to X$ as follows;

$$f(\alpha)=\cap\{U^n_{\alpha_n}/n\in N\}\qquad \text{for } \alpha=(\alpha_1,\alpha_2,\cdots)\in T,$$

then f is clearly a single-valued mapping from T onto X. Since

$$f(N(\alpha_1,\cdots,\alpha_n))=\cap\{U^i_{\alpha_i}/1\leqq i\leqq n\},$$

it follows that f is open. Let p,q be distinct points in X; then by Lemma 1 we admit an integer n in N such that q does not belong to $S(p,\mathcal{U}_n)$. In this case it is proved that