131. *On a Theorem of Wallace and Tsushima*

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1. As was pointed out in Math. Reviews 22 (1961), #12146, the proof of [8; Theorem] contains an error but the theorem holds good for solvable groups and groups with p-complement. Recently, Y. Tsushima [6] has showed that the theorem is still true for p-solvable groups. In the present paper, we shall give an alternative proof to the above fact, and several related results on the radical of a group algebra. We are indebted to Mr. Y. Ninomiya and Mr. Y. Tsushima for their useful advice.

Let K be an algebraically closed field of characteristic $p>0$, and G a finite group with a normal subgroup N such that $|N|$ is not a power of p and G/N is a p-group. Further, G_p will represent a p-Sylow subgroup of G, KG the group algebra of G over K, $J(KG)$ the radical of KG, and $[J(KG):K]$ the K-dimension of $J(KG)$.

2. Now, let $\{T_1, T_2, \cdots, T_s\}$ be the set of all non-conjugate irreducible KN-modules, and G_i the inertia group of T_i, where T_1 corresponds to the 1-representation. By [4; (III. 3.1)] each T_i can be extended uniquely to an irreducible module $\hat{T}_i$ of G_i. We shall prove first the following:

Lemma 1 (cf. [2, (50.2)]). *$\{\hat{T}_1^G, \hat{T}_2^G, \cdots, \hat{T}_s^G\}$ is the set of all irreducible modules of G.*

Proof. At first, we shall show that $\hat{T}_i^G$ is irreducible (cf. [5, Lemma 2]). Let M be a maximal KG-submodule of $\hat{T}_i^G$. By $\mathrm{Hom}_{KG_i}(\hat{T}_i, \hat{T}_i^G/M) \cong \mathrm{Hom}_{KG}(\hat{T}_i^G, \hat{T}_i^G/M) \neq 0$, there exists a KG_i-submodule S_i of $\hat{T}_i^G/M$, which is KG_i-isomorphic to $\hat{T}_i$. By Clliford's theorem, $\hat{T}_i^G/M$ is KN-isomorphic to a direct sum of e-copies of $\sum_{r=1}^t \oplus T_i^{(x_r)}$, where $\{x_r\}$ is a left cross section of G_i in G. Therefore, $(G:G_i)[T_i:K]=[\hat{T}_i^G:K] \geqq [\hat{T}_i^G/M:K]=e(G:G_i)[T_i:K]$ and $[\hat{T}_i^G:K]=[\hat{T}_i^G/M:K]$, which means that $M=0$ and $\hat{T}_i^G$ is irreducible. Next, we shall prove that the above modules are all non-isomorphic. Let $\{y_l \mid 1 \leqq l \leqq r\}$ is a left cross section of G_j in G. Then $\mathrm{Hom}_{KG}(\hat{T}_i^G, \hat{T}_j^G) \cong \mathrm{Hom}_{KG_i}(\hat{T}_i, \hat{T}_j^G) \subseteq \mathrm{Hom}_{KN}(T_i, \hat{T}_j^G) \cong \sum_{l=1}^r \oplus \mathrm{Hom}_{KN}(T_i, y_l \otimes T_j)=0$ for $i \neq j$. Hence, it remains only to prove that s is the number of p-regular classes of G. Let $\{S_1, S_2, \cdots, S_k\}$ be the set of all irreducible representations of N, ω_i Brauer character of S_i. Then ω_i is conjugate to ω_j if and only if S_i is conjugate to S_j. By Brauer's permutation lemma [3, (12.1)], the number of orbits of a