130. On C. Loncour's Results

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The purpose of this note is to present a theorem which refines the results of C. Loncour [3, Theorems 5 and 6] with a simple proof. Let G be the direct product of finite groups M and M', KG the group algebra of G over a field K of characteristic p, $J(KG)$ the radical of KG, $t(G)$ the nilpotency index of $J(KG)$. Let $r_i=[J(KM)^i:K]$ (the K-dimension of $J(KM)^i$) and $r'_i=[J(KM')^i:K]$, where $J(KM)^0=KM$ and $J(KM')^0=KM'$. Let s be a fixed integer such that $1\leqq s<t(G)$. Let $T_i=\{a_{ij}\mid 1\leqq j\leqq r_i-r_{i+1}\}$ be a subset of $J(KM)^i$ which forms a K-basis of $J(KM)^i$ modulo $J(KM)^{i+1}$ (for $i<s$), and $T_s=\{a_{sk}\mid 1\leqq k\leqq r_s\}$[1] a K-basis of $J(KM)^s$. Quite similarly, we define $T'_k=\{a'_{kl}\mid 1\leqq l\leqq r'_k-r'_{k+1}\}$ and $T'_s=\{a'_{sl}\mid 1\leqq l\leqq r'_s\}$ for KM'.

Now, our theorem is stated as follows:

Theorem.[2] (1) $[J(KG)^s:K]=\sum_{i=0}^s r_i r'_{s-i}-\sum_{i=1}^s r_i r'_{s-i+1}$.

(2) $J(KG)^s=\sum_{i=0}^s J(KM)^i J(KM')^{s-i}$.

(3) $t(G)=t(M)+t(M')-1$.

(4)[3] *$\Omega=\bigcup_{s\leqq i+k} T_i T'_k$ forms a basis for $J(KG)^s$, where* $T_iT'_k=\{tt'\mid t\in T_i, t'\in T'_k\}$.

Proof. (1) and (2): We assume first that $s=1$. Let L be a splitting field for G and a finite dimensional separable extension of K. Then L is a splitting field for M and M', $[J(KG):K]=[J(LG):L]$ (cf. [2, p. 252]), and $\{U_i\otimes V_j\mid 1\leqq i\leqq a, 1\leqq j\leqq b\}$ is the set of all irreducible LG-modules (cf. [1, p. 586]), where $\{U_i\mid 1\leqq i\leqq a\}$ and $\{V_j\mid 1\leqq j\leqq b\}$ are the sets of all irreducible LM-modules and LM'-modules, respectively. Thus, $r_1r'_0+r_0r'_1-r_1r'_1=r'_0(r_0-\sum_{i=1}^a[U_i:L]^2)+r_0(r'_0-\sum_{i=1}^b[V_i:L]^2)-(r_0-\sum_{i=1}^a[U_i:L]^2)(r'_0-\sum_{i=1}^b[V_i:L]^2)=r_0r'_0-\sum_{i=1}^a\sum_{j=1}^b[U_i:L]^2[V_j:L]^2=[J(LG):L]=[J(KG):K]$, which proves (1) for the case $s=1$. Noting that $J(KM)M'\cap J(KM')M=J(KM)J(KM')\cong J(KM)\otimes J(KM')$, we can see that $[(J(KM)M'+J(KM')M):K]=[J(KM)M':K]+[J(KM')M:K]-[(J(KM)M'\cap J(KM')M):K]=r_1r'_0+r_0r'_1-r_1r'_1=[J(KG):K]$. Since $J(KM)M'+J(KM')M$ is a nilpotent ideal whose K-dimension is $[J(KG):K]$, we have $J(KG)=J(KM)M'+J(KM')M$, which proves

1) If $J(KM)^t=0$ for some $t<s$, then we set $T_j=\phi$ and $r_j=0$ for $j\geqq t$.

2) Cf. [2, pp. 122–123 and 251–254].

3) Ω is slightly different from C. Loncour's basis of [3, Theorem 5 (2)]. However, it is easy to give his basis for $J(KG)$ by Theorem (4).