122. The Fixed Point Set of an Involution and Theorems of the Borsuk-Ulam Type

By Akio HATTORI
Department of Mathematics, University of Tokyo

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1. Statement of results. In this note, h^* will denote either the unoriented cobordism theory $\mathfrak{N}^*$ or the usual cohomology theory with Z_2-coefficients $H^*(\ ;Z_2)$. The corresponding equivariant cohomology theory for Z_2-spaces will be denoted by $h^*_{Z_2}$.

Let M be a manifold and σ an involution on M.[1] We define an embedding $\varDelta: M \to M^2 = M\times M$ by $\varDelta(x)=(x,\sigma x)$. Then $\varDelta$ is equivariant with respect to the involution σ on M and the involution T on M^2 which is defined by $T(x_1,x_2)=(x_2,x_1)$. Let $\varDelta_!: h^q_{Z_2}(M)\to h^{q+m}_{Z_2}(M^2)$ denote the Gysin homomorphism for $\varDelta$, where $m=\dim M$. We put $\theta(\sigma)=\varDelta_!(1) \in h^m_{Z_2}(M^2)$.

In the present note we shall give an explicit formula for $\theta(\sigma)$ and apply it to get theorems of the Borsuk-Ulam type. Our results generalize those of Nakaoka [3], [4]. From the formula for $\theta(\sigma)$ we shall also derive a sort of integrality theorem concernining the fixed point set of σ; see Theorem 4. Detailed accounts will appear elsewhere.

Let S^∞ be the infinite dimensional sphere with the antipodal involution. The projection $\pi: S^\infty\times M^2 \to S^\infty \times_{Z_2} M^2$ induces the Gysin homomorphism $\pi_!: h^*(M^2)\to h^*_{Z_2}(M^2)$ and the usual homomorphism $\pi^*: h^*_{Z_2}(M^2) \to h^*(M^2)$. Let $d: M\to M^2$ be the diagonal map. Since $d(M)$ is the fixed point set of T, $h^*_{Z_2}(d(M))$ is isomorphic to $h^*_{Z_2}(pt)\otimes_{h^*(pt)} h^*(M)$ and d induces $d^*: h^*_{Z_2}(M^2)\to h^*_{Z_2}(pt)\otimes_{h^*(pt)} h^*(M)$.

Lemma 1. *The homomorphism*

$$\pi^*\oplus d^*: h^*_{Z_2}(M^2)\to h^*(M^2)\oplus(h^*_{Z_2}(pt)\otimes_{h^*(pt)} h^*(M))$$

is injective.

We denote by S the multiplicative set $\{w_1^k \mid k\geq 1\}$ in $h^*_{Z_2}(pt)=h^*(P^\infty)$ where w_1 is the universal first Stiefel-Whitney class. If X is a Z_2-space then $h^*_{Z_2}(X)$ is an $h^*_{Z_2}(pt)$-module and we can consider the localized ring $S^{-1}h^*_{Z_2}(X)$ of $h^*_{Z_2}(X)$ with respect to S. Note that $h^*_{Z_2}(pt)$ is isomorphic to a formal power series ring $h^*(pt)[[w_1]]$ and $h^*_{Z_2}(pt)\otimes_{h^*(pt)} h^*(M)$

1) In this note we work in the smooth category. All manifolds will be connected, compact and without boundary unless otherwise stated.