166. *Note on Approximation of Nonlinear Semigroups*

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Let X be a Banach space with a norm $|\ \ |$, and let $\{T(t);\, t\geqq 0\}$ be a *contraction* (*nonlinear*) *semigroup* on a closed *convex* subset C of X, namely a family of operators from C into C satisfying the following conditions:

(i) $T(0)=I$ (the identity), $T(t+s)=T(t)T(s)$ for $t, s\geqq 0$;

(ii) $|T(t)x-T(t)y|\leqq|x-y|$ for $t\geqq 0$ and $x, y\in C$;

(iii) $\lim_{t\downarrow 0} T(t)x=x$ for $x\in C$.

For each $\lambda, h>0$ we define

$$A_h=h^{-1}(T(h)-I) \quad\text{and}\quad J_{\lambda,h}=(I-\lambda A_h)^{-1}.$$

It is well known that $J_{\lambda,h}$ is a contraction operator from C into C and that A_h generates a unique contraction semigroup $\{T_h(t);\, t\geqq 0\}$ on C such that $(d/dt)T_h(t)x=A_hT_h(t)x$ for $x\in C$ and $t\geqq 0$ (e.g. see [1] and [3]). The purpose of the present note is to prove the following

Theorem. *For each $x\in C$, we have*

(a) $T(t)x=\lim_{h\downarrow 0} T_h(t)x$

uniformly on every bounded interval of $[0,\infty)$,

(b) $T(t)x=\lim_{h\downarrow 0}\{(1-t)I+tT(h)\}^{[1/h]}x$

uniformly in $t\in[0,1]$, *and*

(c) $T(t)x=\lim_{(\lambda,h)\to(0,0)}(I-\lambda A_h)^{-[t/\lambda]}x$

uniformly on every bounded interval of $[0,\infty)$, *where* [ ] *denotes the Gaussian bracket.*

Remark. These results were obtained for $x\in\bar{E}$ by I. Miyadera [3], where $E=\{x\in C;\, |A_hx|=O(1) \text{ as } h\downarrow 0\}$. Recently Y. Kobayashi [2] showed that (a) holds true for $x\in C$ by using an advanced convergence theorem.

We now set for $t>0$ and $x\in C$

$$\gamma(t)=8\cdot\sup\{|T(\eta)x-x|;\, 0\leqq\eta\leqq t\}.$$

Clearly $\gamma(t)$ is non-decreasing and $\gamma(t)\downarrow 0$ as $t\downarrow 0$ by (iii). The following lemma is in Crandall-Liggett [1; Lemma 3.3].

Lemma. *For $x\in C$ and $\delta>0$*

$$|J_{\lambda,h}x-x|\leqq\gamma(2\delta) \qquad \text{if } \lambda, h<\delta.$$

To prove Theorem we start from the following inequalities which are found in [3; (3.4), (3.6) and one in p. 257]: