165. On Approximation of Nonlinear Semi-groups

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1. Let X be a real Banach space and let X_0 be a subset of X. By a *contraction semi-group* on X_0, we mean a family $\{T(t); t\geq 0\}$ of operators from X_0 into itself satisfying the following conditions:

(i) $T(0)=I$ (the identity), $T(t+s)=T(t)T(s)$ for $t, s\geq 0$;

(ii) $\|T(t)x-T(t)y\|\leq\|x-y\|$ for $t\geq 0$ and $x, y\in X_0$;

(iii) $\lim_{t\to 0+} T(t)x=x$ for $x\in X_0$.

We define the *infinitesimal generator* A_0 of $\{T(t); t\geq 0\}$ by $A_0x=\lim_{h\to 0+} h^{-1}(T(h)x-x)$, whenever the right side exists.

Throughout this paper, we assume that X_0 is a closed convex set in X and $\{T(t); t\geq 0\}$ is a contraction semi-group on X_0. Let us set

(1.1) $$A_h=h^{-1}(T(h)-I) \qquad \text{for } h>0.$$

Then, for each h, there is the unique contraction semi-group $\{T_h(t); t\geq 0\}$ on X_0, with the infinitesimal generator A_h, and it satisfies

(1.2) $$(d/dt)T_h(t)x=A_hT_h(t)x \qquad \text{for } x\in X_0 \text{ and } t\geq 0.$$

(See Appendix in [10].)

Our purpose is to prove the following theorem.

Theorem. *For each $x\in X_0$, we have*

(1.3) $$T(t)x=\lim_{h\to 0+} T_h(t)x \qquad \textit{for } t\geq 0,$$

and the convergence is uniform with respect to t in every bounded interval of $[0, \infty)$.

Remarks. 1) I. Miyadera showed in [9] that the convergence (1.3) holds true for $x\in\bar{E}$, where E is the set of $x\in X_0$ such that $\|A_hx\|$ is bounded as $h\to 0+$. Under the similar conditions, many authors have also treated the convergence (1.3). (See [2], [4], [8] and [10].)

2) This theorem is well known in linear theory. (See [5].)

2. For the proof of Theorem, we shall prepare several lemmas in this section. The following is known.

Lemma 1. *Let $x\in X_0$ and $h>0$. Then for $t>0$,*

(2.1) $$\|A_hT_h(t)x\|\leq\|A_hx\|,$$

(2.2) $$\|T_h(t)x-x\|\leq t\|A_hx\|.$$

Let F be the duality map on X into X^* and we set $\langle x, y\rangle_s=\sup\{\langle x, f\rangle; f\in F(y)\}$ for $x, y\in X$.

Lemma 2. *Let $x, z\in X_0$, $h>0$ and n be a positive integer. Then we have*

(2.3) $$\|z-x\|^2\geq\|T(nh)z-x\|^2+2\sum_{i=1}^n h\langle -A_hx, T(ih)z-x\rangle_s.$$