157. On the Trotter-Lie Product Formula*)

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1. In [1, Proposition 7.9] Chernoff gives an example of a pair A, B of nonnegative selfadjoint operators such that

(1) $$(e^{-tA/n}e^{-tB/n})^n \xrightarrow[s]{} 0 \quad \text{as } n\to\infty,\ t>0,$$

where $\xrightarrow[s]{}$ denotes strong convergence. In this example, A is a differential operator of common type while B is an operator of multiplication with a highly singular function; the proof makes essential use of the Wiener integral.

In what follows we shall show that if A, B are nonnegative selfadjoint, (1) is true whenever $\boldsymbol{D}(A^{1/2})\cap \boldsymbol{D}(B^{1/2})=\{0\}$, which is the case in Chernoff's example. [$\boldsymbol{D}(T)$ denotes the domain of T.] Furthermore, we shall show that (1) is true in the general case if applied to a vector orthogonal to $\boldsymbol{D}(A^{1/2})\cap \boldsymbol{D}(B^{1/2})$.

We shall consider this problem for a more general sequence

(2) $$U_n(t)=[f(tA/n)g(tB/n)]^n, \qquad n=1,2,\cdots,$$

where f, g are taken from the class of real-valued, Borel measurable functions ϕ on $[0,\infty)$ such that

(3) $$0<\phi(t)\leq 1, \quad \phi(0)=1, \quad \phi'(0)=-1.$$

$\phi(t)=e^{-t}$ belongs to this class. Another example is $\phi(t)=(1+t)^{-1}$, which is perhaps more important in connection with approximation theory in differential equations.

We note that (3) already implies that

(4) $$\phi(tA)\xrightarrow[s]{} 1, \qquad t\downarrow 0,$$

whenever A is nonnegative selfadjoint.

To prove our results, we need a mild additional condition for at least one of f and g, namely

(5) $t^{-1}[1-\phi(t)]$ is monotone nonincreasing on $0<t<\infty$.

Note that (5) is again satisfied by $\phi(t)=e^{-t}$ and $(1+t)^{-1}$.

We can now state our main theorem.

Theorem 1. *Let A, B be nonnegative selfadjoint operators in a Hilbert space $\boldsymbol{H}$. Assume that both f and g satisfy* (3) *and at least one of them satisfies* (5). *If $v\in \boldsymbol{H}$ is orthogonal to $\boldsymbol{D}(A^{1/2})\cap \boldsymbol{D}(B^{1/2})$, then $U_n(t)v\to 0$ as $n\to\infty$, uniformly on compact sets of $t>0$.*

Theorem 1 raises the question as to what happens to $U_n(t)v$ if

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