154. *Fricke Formula for Quaternian Groups*

By Nobuel Ishii

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For a square free positive integer N, let $\Gamma_0(N)$ be the congruence subgroup of level N, i.e.

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z})\,;\, C\equiv 0 \text{ mod. } N\right\} \quad \text{and} \quad \Gamma_0^*(N)$$

be the group generated by $\Gamma_0(N)$ and the element $x=\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. Fricke (Die Elliptischen Funktionen und ihre Anwendungen II pp. 357–367) has given a following relation between the genus g of the Riemann surface obtained from $\Gamma_0(N)$ and the genus g^* of that of $\Gamma_0^*(N)$ for $N>4$:

$$2g^*-g=1-\frac{1}{2}\delta_N h(-4N)$$

where $h(-4N)$ is the class number of the order of $\boldsymbol{Q}(\sqrt{-N})$ with discriminant $-4N$ and $\delta_N=2, 4/3, 1$ for $N\equiv 7$, $N\equiv 3$, otherwise, mod. 8, respectively. In this note, we shall give a similar formula for some arithmetic Fuchsian group $\bar{\Gamma}$ obtained from an indefinite quaternion algebra and a certain normalizer $\bar{\Gamma}^*$ of $\bar{\Gamma}$ with $[\bar{\Gamma}^*:\bar{\Gamma}]=2$. To be more precise, let B be a quaternion algebra over a totally real algebraic number field k and let R be an order of square free stufe (cf. [1]). Let v be a finite place of k where the completion R_v is not isomorphic to the total matrix ring with integral coefficients. If the class number of R is one, for such v, there exists an element π_v of B such that π_v is a prime element of R_v and is a unit at any other places. Now we take Γ=the group of totally positive units in R, and Γ^*=the group generated by Γ and π_v (or product of such π_v's). Let $\bar{\Gamma}$ (resp. $\bar{\Gamma}^*$) denotes the Fuchsian group corresponding to Γ (resp. Γ^*). Then, denoting by g (resp. g^*) the genus of $\bar{\Gamma}$ (resp. $\bar{\Gamma}^*$), we have the formula (Corollary to Theorem 3.0) of the form;

$2g^*-g=$(*sum of class numbers of certain totally imaginary quadratic extensions of* k).

Our proof depends on the well known *Hurwitz formula* which has the following form under our assumption that $[\bar{\Gamma}^*:\bar{\Gamma}]=2$:

$2g-2=2(2g^*-2)+$(*the number of ramified fixed points of* Γ)

(see, for example, G. Shimura: Introduction to the arithmetic theory of automorphic functions. Iwanami Shoten, 1971, p. 19). Thus our problem amounts to determine the conjugate classes of elliptic points