178. *Lipschitz Functions and Convolution*

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1. Introduction. In this paper we shall consider functions defined on the torus. S. Bernstein's theorem [7; vol. 1, p. 240] says that the set Lip α is contained in the space A of functions with an absolutely convergent Fourier series when $\alpha > 1/2$. As is well known, the space A coincides with the space $L^2 * L^2$ [7; vol. 1, p. 251]. These assert that Lip α is contained in $L^2 * L^2$ if $\alpha > 1/2$. On the other hand, R. Salem's result [6] implies that the space $L^1 * L^\infty$ is equal to the space C of all continuous functions (see also [2]). Therefore it is trivial that Lip α is contained in $L^1 * L^\infty$ for $\alpha > 0$. Then it is expected that Lip α is contained in $L^p * L^q$ if $\alpha > 1/q$ where $1 < p < 2$ and $1/p + 1/q = 1$. This fact is proved by using results of N. Aronszajn-K. T. Smith and A. P. Calderon (see [3]). We shall give an elementary proof.

Theorem 1. *Let $1 \leqq p < \infty$, $1/p + 1/q = 1$ and $1 \leqq r \leqq \infty$. If $f \in L^r$ and $\|\sigma_n - f\|_r = O(n^{-\alpha})$ for some $\alpha > 1/q$, then $f \in L^p * L^r$ where σ_n is the n-th $(C, 1)$ mean of Fourier series of f.*

Corollary 1. *Let $1 \leqq p \leqq 2$ and $1/p + 1/q = 1$. If $\alpha > 1/q$, then* Lip α *is contained in $L^p * L^q$. There exists however a function which belongs to* Lip $1/q$ *but not to $L^p * L^q$ if $p \neq 1$.*

Now we denote by BV_p the space of functions of p-bounded variation for $1 \leqq p \leqq \infty$ (see [3] or [5] for definition). It is obvious that BV_1 is the set of functions of ordinary bounded variation and BV_∞ is of bounded functions.

Corollary 2. *If $1 \leqq p \leqq 2$ and $1/p + 1/q = 1$, then the intersection of* Lip α *and $BV_{q-\varepsilon}$ is contained in $L^p * L^q$ for $\alpha > 0$ and $\varepsilon > 0$.*

The case $p = 2$ and $\varepsilon = 1$ is A. Zygmund's theorem [7; vol. 1, p. 241] by $A = L^2 * L^2$ and the case $p = 1$, as previously stated, is trivial from R. Salem's result.

In the proof of Theorem 1, we use a method of R. Salem [6].

2. Lemmas. We shall here state some lemmas.

Lemma 1. *Let $1 \leqq p \leqq \infty$ and $1/p + 1/q = 1$. If a positive and convex sequence $\{\lambda_n\}$ tending to zero satisfies the condition*

$$\sum_{n=1}^{\infty} n^{1+1/q} (\lambda_{n-1} + \lambda_{n+1} - 2\lambda_n) < \infty,$$

then there is a function g in L^p such that $\hat{g}(n) = \lambda_{|n|}$ for every integer n.

Proof. Denoting the Fejér kernel by K_n, the series