69. Analytic Functions in a Neighbourhood of Boundary

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Let R be an end of a Riemann surface with compact relative boundary ∂R. Let $F_i(i=1,2,\cdots)$ be a connected compact set such that $F_i \cap F_j=0: i\neq j$, $\{F_i\}$ clusters nowhere in $R+\partial R$ and $R-F(F=\Sigma F_i)$ is connected. We call $R'=R-F$ a lacunary end. If there exists a determining sequence $\{\mathfrak{V}_n(\mathfrak{p})\}$ of a boundary component $\mathfrak{p}$ of R such that $\inf_{z\in\partial\mathfrak{V}_n(\mathfrak{p})} G(z,p_0)>\varepsilon_0>0, n=1,2,\cdots$ and $\partial\mathfrak{V}_n(\mathfrak{p})$ is a dividing cut, we say F is completely thin at $\mathfrak{p}$, where $G(z,p_0)$ is a Green's function of R'. If there exists an analytic function $w=f(z): z\in R'$ such that the spherical area of $f(R')$ is finite over the w-sphere, we say R' satisfies the condition S. If there exists a non const. $w=f(z)$ such that $C(f(R'))$ (complementary set of $f(R')$ with respect to w-sphere) is a set of positive capacity, we say R' satisfies the condition B. Then we proved

Theorem ([1]). *Let R be an end of a Riemann surface $\in 0_g$. If F is completely thin at $\mathfrak{p}$ and $R'=R-F$ satisfies the condition* S, *then the harmonic dimension* (*the number of minimal points of R over $\mathfrak{p}$*)$<\infty$.

In this note we show the above theorem is valid under the condition B instead of the condition S. Since if the spherical area of $f(R')<\infty$, we can find a neighbourhood $\mathfrak{V}_{n_0}(\mathfrak{p})$ of $\mathfrak{p}$ such that $C(f(\mathfrak{V}_{n_0}(\mathfrak{p})\cap R'))$ is a set of positive capacity, the result which will be proved is an extension of the theorem.

Let $R\notin 0_g$ be a Riemann surface. Let $V(z)$ be a positive harmonic function in $R-F$ such that $V(z)=\infty$ on $F, V(z)$ is singular in $R-F$ and $D(\min(M,V(z)))\leqq M\alpha$ for any $M<\infty$, α is a const., we call $V(z)$ a generalized Green's function (abbreviated by G.G.), where F is a set of capacity zero. Then

Lemma 1. 1) *Let $V(z)$ be a* G.G. *in R. Then there exists a cons. α such that $D(\min(M,V(z)))=M\alpha$ and $\int_{C_M}\frac{\partial}{\partial n}V(z)ds=\alpha: C_M=\{z\in R: V(z)=M\}$ for any $M<\infty$.* 2). *Let $G(z,p_i)(i=1,2,\cdots)$ be a Green's function and $\{p_i\}$ be a sequence such that $G(z,p_i)$ converges to $G(z,\{p_i\})$. Then $G(z,p)$ and $G(z,\{p_i\})$ are* G.G.s *such that*

$$\int_{C_M}\frac{\partial}{\partial n}G(z,p)ds=2\pi \quad\text{and}\quad \int_{C_M}\frac{\partial}{\partial n}G(z,\{p_i\})ds\leqq 2\pi. \tag{1}$$

Let $R'=\{z\in R: G(z,p_0)>\delta\}$ and let $\hat{R}'$ be the symmetric image of R'