# 111. *On the σ-Socle of a Module*

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Let $R$ be a ring with identity and let $\sigma$ be a left exact radical on $R$-mod such that $T(\sigma)$ is a TTF class. The purpose of this paper is to show that, for any module $M$, the sum of all $\sigma$-simple submodules of $M$ coincides with the intersection of all $\sigma$-essential submodules of $M$. In case $\sigma=1$, i.e., $T(\sigma)=R$-mod, the above result means the so-called Sandomierski-Kasch's characterization of the socle of a module (see [1, p. 62]).

Let $\sigma$ be a left exact preradical on the category $R$-mod of unital left $R$-modules. Then the class $T(\sigma)=\{M \mid \sigma(M)=M\}$ is closed under submodules, quotients and direct sums. The modules in $T(\sigma)$ are called *σ-torsion*. A submodule $L$ of a module $M$ with $M/L \in T(\sigma)$ is called *σ-open* in $M$. If $L$ is both $\sigma$-open and essential in $M$, we say that $L$ is *σ-essential* in $M$. The *σ-socle* of a module $M \neq 0$, denoted by $\sigma\text{-soc}(M)$, is defined as the intersection of all $\sigma$-essential submodules of $M$. If $M=0$ we define $M=\sigma\text{-soc}(M)$. A module $S$ is called *σ-simple* if for any $\sigma$-open submodule $A$ of $S$, either $A=S$ or $A=0$.

**Lemma.** *If $S$ is a σ-simple submodule of $M$, then* $S\subseteq\sigma\text{-soc}(M)$.

**Proof.** We may assume $S\neq 0$. If $L$ is a $\sigma$-essential submodule of $M, S\cap L\neq 0$ and $S\cap L$ is $\sigma$-open in $S$, since $S/(S\cap L)\cong(S+L)/L\subseteq M/L \in T(\sigma)$. Thus $S\cap L=S$ and so $S\subseteq L$.

A module $M$ is *σ-semisimple* if every $\sigma$-open submodule of $M$ is a direct summand of $M$. From [2], we quote the following facts:

(A) A $\sigma$-torsion module is $\sigma$-semisimple if and only if it is semisimple.

(B) If $M$ is $\sigma$-semisimple, and $N$ is any submodule of $M$, then $M/N$ is $\sigma$-semisimple.

Now we assume moreover that $\sigma$ is a left exact radical such that $T(\sigma)$ is a TTF class, i.e., $T(\sigma)$ is closed additionally under extensions and direct products. In this case, the corresponding topology $\mathcal{F}=\{I \mid I$ is a left ideal with $R/I \in T(\sigma)\}$ has a smallest member $U$. $U$ is idempotent and $T(\sigma)=\{M \mid UM=0\}$.

**Theorem.** *If σ is a left exact radical such that $T(\sigma)$ is a* TTF *class, then for any module $M$,* $\sigma\text{-soc}(M)=\Sigma\{S\subseteq M \mid S$ *is σ-simple*$\}$. *Moreover* $\sigma\text{-soc}(M)$ *is a direct sum of σ-simple submodules.*

**Proof.** We show only the last assertion holds, then the former