107. *Limit Theorems for Poisson Branching Processes*

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1. The process treated here is a model of the population growth in a biological system in which each object gives births at various times of its life length and new born objects behave as their parents independently of others. The process is specified by two nonnegative continuous functions on $[0, \infty)$ $\lambda(x)$, $\mu(x)$ and a probability generating function $h(s)=\sum_{n=1}^{\infty} h_n s^n$, $\sum_{n=1}^{\infty} h_n=1$, $h_n \geqq 0$ $(n=1,2,\cdots)$; *a living object of age x gives births to j objects before it reaches age $x+dx$ without dying itself with a probability $h_j\lambda(x)dx$ and dies before age $x+dx$ with a probability $\mu(x)dx$* where these probabilities are independent of each other and of past history. This process appeared in [2] as a special case of general age dependent branching processes and was called a *Poisson branching process*. In this paper limit theorems will be given for probability generating functions of the population size at time t of Poisson branching processes. Limit theorems of such type are studied by Ryan [5] for subcritical general age dependent branching processes. His results contain a part of ours as a special case. The forms and proofs of theorems given here are simpler than Ryan's and almost parallel with ones of age dependent branching processes given in [1].

2. Let $Z(t)$ be the population size at time t of a Poisson branching process specified by $\lambda(x)$, $\mu(x)$ and $h(s)$ as in the first section and let $F(s,t)$ be its generating function; $F(s,t)=E[s^{Z(t)}]$, $0\leqq s\leqq 1$. We always assume that the process starts with a single object of age 0. Let L be the time when the initial object dies and $G(t)$ be the distribution function of L; $G(t)=\int_0^t \mu(u)\exp\left(-\int_0^u \mu(r)dr\right)du$. By conditioning on L we get

$$F(s,t)=s(1-G(t))E\left[\exp\left\{\int_0^t \log F(s,t-u)dN(u)\right\}\Big| L>t\right]$$
$$+\int_0^t E\left[\exp\left\{\int_0^u \log F(s,t-v)dN(v)\right\}\Big| L=u\right]dG(u),$$

in which we denote by $N(t)$ the number of direct children of the initial particle that have been ever born untill time t. Then we have

$$F(s,t)=s(1-G(t))\exp\left\{\int_0^t (h(F(s,t-u))-1)\lambda(u)du\right\}$$
$$+\int_0^t \exp\left\{\int_0^u (h(F(s,t-v))-1)\lambda(v)dv\right\}dG(u). \tag{1}$$