59. *Invariant Measures for Bounded Amenable Semigroups of Operators*

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(Comm. by Kôsaku YOSIDA, M. J. A., May 12, 1976)

In this paper we consider the invariant measure problem for bounded amenable semigroups of positive L_1-operators. A necessary and sufficient condition is given for the existence of finite equivalent invariant measures for such semigroups.

Let $(X, \mathcal{F}, m)$ be a probability space and let $L_p(X) = L_p(X, \mathcal{F}, m)$, $1 \leqq p \leqq \infty$, be the usual Banach spaces. For a set $A \in \mathcal{F}$, 1_A is the indicator function of A and $L_p(A)$ denotes the Banach space of all $L_p(X)$-functions that vanish on $X - A$. Let $\Gamma = \{T\}$ be a semigroup of positive linear operators on $L_1(X)$. Γ is called *bounded* if $\sup\{\|T\|_1 : T \in \Gamma\} < \infty$. Let $B(\Gamma)$ denote the space of all bounded real-valued functions on Γ. A *mean* φ on $B(\Gamma)$ is a linear functional on $B(\Gamma)$ such that

$$\inf\{b(T) : T \in \Gamma\} \leqq \varphi(b) \leqq \sup\{b(T) : T \in \Gamma\}$$

for all $b \in B(\Gamma)$. A mean φ on $B(\Gamma)$ is *left* [*right*] *invariant* if

$$\varphi({}_T b) = \varphi(b) \qquad [\varphi(b_T) = \varphi(b)]$$

for all $b \in B(\Gamma)$ and $T \in \Gamma$, where ${}_T b$ and b_T are the functions on Γ defined by ${}_T b(S) = b(TS)$ and $b_T(S) = b(ST)$ for all $S \in \Gamma$, respectively. An *invariant mean* is a left and right invariant mean. If $B(\Gamma)$ has a left [right] invariant mean, Γ is called *left* [*right*] *amenable.* If $B(\Gamma)$ has an invariant mean, then Γ is called *amenable.* It is well-known that commutative semigroups, solvable groups, locally finite groups, etc., are amenable (for these and more see Day [1]).

Recently the author [4] has proved that if $\Gamma = \{T\}$ is a bounded *left amenable* semigroup of positive linear operators on $L_1(X)$, then the following two conditions are equivalent: (0) There exists a strictly positive function $f_0 \in L_1(X)$ with $Tf_0 = f_0$ for all $T \in \Gamma$; (i) $A \in \mathcal{F}$ and $m(A) > 0$ imply $\inf\left\{\int_A T1\, dm : T \in \Gamma\right\} > 0$. In the present paper we shall assume that Γ is a bounded *amenable* semigroup of positive linear operators on $L_1(X)$. Let us denote by IM the set of all invariant means on $B(\Gamma)$ and define, for $b \in B(\Gamma)$,

$$M(b) = \sup\{\varphi(b) : \varphi \in IM\}.$$

Then we have the following

Theorem. *Let $\Gamma = \{T\}$ be a bounded amenable semigroup of positive linear operators on $L_1(X)$. Then the following two conditions are*