136. The Concrete Description of the Colocalization

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Introduction. Recently K. Ohtake [5] proved that for a torsion theory $(\mathcal{T}, \mathcal{F})$ there is a colocalization functor if and only if $\mathcal{F}$ is a TTF-class, in this case we have another torsion theory $(\mathcal{F}, \mathcal{D})$ and T. Kato [2], K. Ohtake [5] proved that there is an equivalence between the colocalization subcategory $[C]$ of Mod-R with respect to $(\mathcal{T}, \mathcal{F})$ and the localization subcategory $[L]$ of Mod-R with respect to $(\mathcal{F}, \mathcal{D})$.

In this paper, we shall show a colocalization of any module M_R can be obtained by $M\otimes_R I\otimes_R I$ concretely where I is a corresponding two sided ideal, i.e. the unique minimal ideal belonging to the filter which corresponds to $(\mathcal{F}, \mathcal{D})$.

As an application of this, we get self-contained and fairly simple proofs of the results in [5].

The concrete description of the colocalization. Throughout this paper, ring R means an associative ring with unit, Mod-R (resp. R-Mod) denotes a class of all unital right (resp. left) R-modules and $(\mathcal{A}, \mathcal{B})$ (resp. $(\mathcal{A}^*, \mathcal{B}^*)$) denotes a torsion theory in Mod-R (resp. R-Mod), about which the reader is referred to [6].

Let $(\mathcal{A}, \mathcal{B})$ be a torsion theory. A module M_R is called "*divisible*" if $Ext_R^1(K, M)=0$ for any $K\in\mathcal{A}$, dually "*codivisible*" if $Ext_R^1(M, K)=0$ for any $K\in\mathcal{B}$, and a map $M_R\xrightarrow{f}L(M)_R$ (resp. $C(M)_R\xrightarrow{f}M_R$) is called "*localization*" of M_R (resp. "*colocalization*" of M_R) if $\ker(f)$, $\mathrm{cok}(f)\in\mathcal{A}$, $L(M)_R\in\mathcal{B}$ and $L(M)$ is divisible (resp. $\ker(f)\in\mathcal{B}$, $\mathrm{cok}(f)\in\mathcal{B}$, $C(M)\in\mathcal{A}$ and $C(M)$ is codivisible).

$[L]$, $[C]$ denote the full subcategory of torsion-free divisible modules in Mod-R and torsion codivisible modules in Mod-R which are called localization subcategory and colocalization subcategory with respect to $(\mathcal{A}, \mathcal{B})$ respectively.

Let I be a two sided idempotent ideal and $\mathcal{F}=\{M_R\in\text{Mod-}R \mid M\cdot I=0\}$, then $\mathcal{F}$ is TTF-class in the sense of Jans [1]. (i.e. closed under taking submodules, extensions and direct products). Any TTF-class in Mod-R is obtained as above, in this case corresponding torsion class and torsion-free class are $\mathcal{T}=\{M_R \mid M\cdot I=M\}$ and $\mathcal{D}=\{M_R \mid \mathrm{Ann}_M(I)=0\}$ respectively. (i.e. $(\mathcal{T}, \mathcal{F})$, $(\mathcal{F}, \mathcal{D})$ are torsion theories.) The corresponding filter with respect to $(\mathcal{F}, \mathcal{D})$ is $\mathcal{J}=\{J_R \mid J_R$ is a right ideal which