147. *The Finite Hilbert Transform on $L_2(0,\pi)$ is a Shift*

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Let v be the finite Hilbert transform on $L_2(0,\pi)$ defined by

$$(V\varphi)(t)=\frac{1}{\pi i}\int_0^\pi \frac{\sin s}{\cos t-\cos s}\varphi(s)ds,$$

where the integral is the Cauchy principal value. In contrast with the development of the spectral theory of a finite Hilbert transform A of the form

$$(Af)(x)=\frac{1}{\pi i}\int_a^b \frac{f(y)}{x-y}dy$$

acting on $L_2(a,b)$, which occurs in airfoil theory, the singular integral operator V on $L_2(0,\pi)$ has not received much attention, while it plays an important role in the theory of singular integral equations (cf. [3]). Let $\varphi_n(t)=\sin nt\ (n=1,2,\cdots)$ and $\psi_n(t)=\cos nt\ (n=0,1,2,\cdots)$. Then the sequences $\{\varphi_n\}$ and $\{\psi_n\}$ of vectors are both orthogonal bases in $L_2(0,\pi)$ and as is seen in Hochstadt [3; p. 160], V is an isometry such that

$$V\varphi_n=-i\psi_n \qquad (n=1,2,\cdots).$$

The first object of this paper is to prove the following decisive result:

Theorem. *The finite Hilbert transform V on $L_2(0,\pi)$ is a unilateral shift of multiplicity* 1.

Next we shall indicate that this result actually offers a new technique in the spectral representation theory for the airfoil operator A and enables us to remove somewhat complicated integral calculations involved in the conventional treatments [4] and [7].

1. The proof of the theorem is done independently of the airfoil operator on $L_2(-1,1)$. First observe that the operator V is symmetrizable in the sense of P. Lax [5] (for symmetrizable operators, see also [1] and [9]). Indeed, for a pair of vectors φ,ψ in $L_2(0,\pi)$, we define a new inner product $(\ ,\ )$ by

$$(\varphi,\psi)=\int_0^\pi \varphi(t)\bar{\psi}(t)\sin t dt.$$

Then it is obviously bounded on $L_2(0,\pi)$ and from the behavior of V on the basis $\{\varphi_n\}$ it is straightforward to verify that

$$(V\varphi_n,\varphi_m)-(\varphi_n,V\varphi_m)=-i\int_0^\pi \sin(m+n)t\sin t dt=0$$

for every n,m. It follows immediately from this that V is self-adjoint