34. On a Property of Transcendental Integral Functions.

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Mr. Tsuji[1] proved that for a class of integral functions $f(z)$, for which $f(0) = a$, $f(z_i) = b$, $(i = 1,2,\cdots)$, where $a \neq b$, $a \neq 0, \neq 1$, and $b \neq 0, \neq 1$ and $|z_1| \leqq |z_2| \leqq \cdots \to \infty$, there exists an infinite number of concentric ring-regions $|z| < R_1$, $R_i < |z| < R_{i+1}$ $(i = 1,2,\cdots)$, R_i depending only on the class, in which all the functions of the class take at least once the value 1 or 0.

We will here prove the following allied

Theorem: *Consider a class of integral functions*

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_m z^m + \cdots\cdots\cdots, \tag{1}$$

for which $|c_m| \geqq \frac{l_0}{m!} > 0$ *for a certain value of* $m \geqq 1$, *and* $|f(z_i)| = l_i < M$ $(i = 1,2,\cdots)$, *where* l_i *are positive constants*[2] *and* $|z_1| \leqq |z_2| \leqq \cdots \to \infty$, *then there exists an infinite number of concentric ring-regions* $|z| < R_1$, $R_i < |z| < R_{i+1}$, $(i = 1,2,\cdots)$, R_i *depending only on the class, in which any function* (1) *takes at least once the value* 1 *or* 0, *and we can find an expression for an infinite number of radii* R_i *of the ring-regions* $R_i < |z| < R_{i+1}$.

Proof. Suppose, if possible, that a function (1) does not take the values 1 and 0 in the ring-region $0 \leqq R_0 < |z| < R$, $R = 2(r_i - R_0) + R_0$, where $|z_i| = r_i$, and therefore in the circle of radius $r_i - R_0$ with center at z_i, then by Landau's theorem[3] we have in $|z - z_i| < \frac{r_i - R_0}{2}$

$$|f(z)| < \Omega(M). \tag{2}$$

Now take $2q\left(q < \left[\frac{2\pi}{1 - R_0/r_i}\right] + 1\right)$ circles $C_{i,\pm h}$ $(h = 1,2,\cdots q)$ of radius

1) Proc. Imperial Academy, 2 (1926) 364–365.
2) In this case it is not necessary that $c_m \neq l_i$.
3) Götting. Nachr. (1910), 309.