33. On an Extension of Pólya's "Ganzwertige ganze Funktion".

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Mr. Pólya treated the integral functions $g(z)$ which take integral values for all integral values of z and called them " ganzwertige ganze Funktionen "[1]. I have tried to extend this idea in the following way:

1. Let us consider a set of positive integers

$$Z: \quad (z_1, z_2, \cdots\cdots\cdots)$$

and a function $g(z)$, which takes integral values (in the rational corpus or imaginary quadratic corpus) for all z_i. Denote by $\pi(n)$ the number of z_i's which is not greater than n and put $\underset{|z|\leqq r}{\text{Max}}\, g(z) = M(r)$. We construct a function $\Psi(x)$, which coincides with $\pi(x)$ for all integral values of x and is otherwise linear in x. With this $\Psi(x)$ we form also a function $\varphi(x)$, which is continuously differentiable and such that $\varphi(0) = 0$ and $\varphi(x) \leqq \Psi(x)$ for $x > 0$. Then we have:

Theorem A. *If we can chose a real function $\rho = \rho(r)$ such that*

$$(r+1)\log r - r - \int_0^r \varphi'(x)\log(\rho - x)dx + \log \rho M(\rho) \longrightarrow -\infty$$

and $$\int_0^r \frac{\varphi(x)}{1+x}dx - \log M(r) \longrightarrow +\infty \qquad as \quad r \longrightarrow +\infty$$

then $g(z)$ must be a polynomial.

We can prove this theorem by means of a method, similar to Pólya's, save as we have to evaluate a quantity of the form

$$\prod_{i=1}^{n}(z - z_i) \quad \text{for} \quad |z| = r.$$

2. As the special cases of this theorem we have:

If one of the following conditions is satisfied:

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1) Pólya, Über die ganzwertige ganze Funktionen, Rend. Palermo, 40 (1915), 1–16. For the literature see my paper under the same title, Tohoku mathematical Journal 27 (1926), 41–52.