32. *Note on a Theorem of Fekete.*

By Buchin Su.

Mathematical Institute, Tohoku Imp. University, Sendai.

(Rec. Feb. 15, 1927. Comm. by M. Fujiwara, M. I. A., March 12, 1927.)

1. Fekete[1] and Bálint[2] proved the following theorem:

If

$$P(z) = p_0 + p_1 z^{\mu_1} + p_2 z^{\mu_2} + \cdots\cdots\cdots\cdots + p_k z^{\mu_k}$$

be a polynomial with $k+1$ terms ($p_0, p_1 \cdots\cdots, p_k$ are any complex numbers other than zero; and $\mu_1, \mu_2, \cdots\cdots, \mu_k$ are integers such that $1 \leqq \mu_1 < \mu_2 < \cdots\cdots < \mu_k$), and $P(-1) \neq P(+1)$, then there exists at least one point z in the circle $|z| \leqq 2 \cdot k \cot \frac{\Phi}{2} \left(\Phi \leqq \frac{\pi}{2}\right)$ in which $P(z)$ takes any given value γ in the domain K', whose boundary consists of two circular arcs subtending an angle Φ to the segment joining the points $P(-1)$ and $P(+1)$.

We can, however, extend this domain for γ into the circle K with centre $\{P(-1)+P(+1)\}/2$ and radius $\left\{ |P(+1)-P(-1)| \cot \frac{\Phi}{2} \right\}/2$, which contains K'.

Our theorem runs as follows:

Theorem 1. *Let $P(-1) \neq P(+1)$, and γ be any point in the circle K with centre $\{P(-1)+P(+1)\}/2$ and radius $\frac{1}{2}|P(+1)-P(-1)| \cot \frac{\Phi}{2}$, where $\Phi \leqq \frac{\pi}{2}$. Then there exists at least one point z in the circle $|z| \leqq 2\,k \cot \frac{\Phi}{2}$, in which $P(z)$ takes the value γ.*

Proof. Draw two circular arcs passing through the points $P(-1)$, $P(+1)$, subtending an angle $\Phi \leqq \frac{\pi}{2}$. Let AA', BB' be the common tangents of two circles and O the midpoint of $M(P(-1))$ $N(P(+1))$. Take a point Q on AA' and a point $R(\gamma)$ on the line OQ. Then since we have

1) Fekete, Jahrsb. d. Deutsch. Math. Ver. 32 (1923), 299-306.
2) Bálint, The same Journal, 34 (1926), 233-237.