27. *On a Property of the Fourier Series of an Almost Periodic Function.*

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In the theory of an analytic almost periodic function Bohr's "Randwertsatz" plays an important rôle. It concerns with an analytic almost periodic function whose Dirichlet exponents have the same sign. In this paper, we shall prove a theorem which is regarded as an extension of Randwertsatz to the case where Dirichlet exponents have not the same sign.[1]

Theorem. If an indefinite integral of an almost periodic function of a real variable

$$f(x)\sim\sum A_n e^{i\Lambda_n x}\qquad(-\infty<x<\infty)$$

is bounded, then the series

$$\sum A_n \operatorname{sgn}\Lambda_n\cdot e^{-\sigma|\Lambda_n|}e^{i\Lambda_n x},$$

where σ is any positive number, is the Fourier series of an almost periodic function.

For the proof,[2] consider the function

$$\varphi_A(t)=\frac{1}{\pi}\int_0^A\frac{xf(x+t)-xf(-x+t)}{\sigma^2+x^2}dx\qquad(-\infty<t<\infty),$$

where A and σ are any positive numbers. Then $\varphi_A(t)$ is an almost periodic function. Indeed, taking τ as a translation-number of $f(t)$ belonging to ε, i.e.

$$|f(t+\tau)-f(t)|\leqq\varepsilon,\qquad-\infty<t<\infty,$$

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1) In case where Dirichlet exponents have not the same sign, the following theorem is known:

If $\qquad f(s)\sim\Sigma A_n e^{\Lambda_n s},\qquad s=\sigma+it$

is almost periodic in $<\alpha,\beta>$ and if its integral $F(s)$ is also almost periodic in $<\alpha,\beta>$ (which is true if $F(s)$ is bounded), then the series

$$\sum_{\Lambda_n<0}A_n e^{\Lambda_n s},\qquad\sum_{\Lambda_n>0}A_n e^{\Lambda_n s}$$

are the Dirichlet series of two functions $f_1(s)$, almost periodic in $<\alpha,+\infty)$ and $f_2(s)$, almost periodic in $(-\infty,\beta>$.

2) I owe this method of proof to Mr. Favard's paper: Sur la fonction conjuguée d'une fonction presque-périodique, Matematisk Tidsskrift (1934), p. 57.