111. A Note on the Continuous Representation of Topological Groups.

By Kôsaku YOSIDA.

Mathematical Institute, Osaka Imperial University.

(Comm. by T. YOSIE, M.I.A., Dec. 12, 1936.)

§ I. In a recent paper,[1] the author treated the group $\mathfrak{D}$ embedded in the *metrical complete ring* $\mathfrak{R}$. Such a group $\mathfrak{D}$ is, by [I], a *Lie group* (as defined in [I][2]) if and only if $\mathfrak{D}$ is locally compact.

In another paper,[3] I obtained a result saying that, if $\mathfrak{D}$ is continuously homomorphic to a connected and locally bicompact topological group $\mathfrak{G}$ (*without any countability axiom*), then $\mathfrak{D}$ is a *Lie representation* (defined in [II]).

A Lie representation $\mathfrak{D}$ is not necessarily a Lie group as defined in [I] (see [II]), though the *infinitesimal operators* of $\mathfrak{D}$ obey the customary rule of the ordinary *Lie-ring*, when $\mathfrak{G}$ is a Lie group (see [II]).

However we may add the following remark:

The representation $\mathfrak{D}$ of $\mathfrak{G}$ is a Lie group (as defined in [I]) *if and only if the homomorphic mapping $\mathfrak{G}\to\mathfrak{D}$ is open.*

Here a continuous mapping is called *open* if the mapped image of any open set is an open set.

Proof. $\mathfrak{D}$ is isomorphic to the quotient group $\mathfrak{G}/\mathfrak{N}$, where $\mathfrak{N}$ is an invariant subgroup closed in $\mathfrak{G}$. We call any set in $\mathfrak{G}/\mathfrak{N}$ open if and only if it corresponds to an open set in $\mathfrak{G}$ by the homomorphic mapping $\mathfrak{G}\to\mathfrak{G}/\mathfrak{N}$. Then $\mathfrak{G}/\mathfrak{N}$ is connected and locally bicompact with $\mathfrak{G}$. Thus $\mathfrak{D}$ is continuously isomorphic to $\mathfrak{G}/\mathfrak{N}$ and the mapping $\mathfrak{G}\to\mathfrak{D}$ is open if and only if the mapping $\mathfrak{G}/\mathfrak{N}\to\mathfrak{D}$ is open.

Hence we may- and shall- assume that $\mathfrak{D}$ is continuously isomorphic to $\mathfrak{G}$.

Thus if the mapping $\mathfrak{G}\to\mathfrak{D}$ is open, $\mathfrak{G}$ and $\mathfrak{D}$ are homeomorphic with each other, and hence $\mathfrak{D}$ is locally bicompact and connected with $\mathfrak{G}$. This proves the sufficiency of the condition of the remark.

As the group $\mathfrak{D}$ is embedded in $\mathfrak{R}$, $\mathfrak{D}$ does not contain an arbitrarily small cyclic subgroup ($\neq$ identical group[4]). $\mathfrak{G}$ enjoys the same property, for $\mathfrak{D}$ is continuousy isomorphic to $\mathfrak{G}$. By a theorem due to A. Komatu and S. Kakutani (see [II]) $\mathfrak{G}$ satisfies the first axiom of countability, since $\mathfrak{G}$ is locally bicompact. Hence $\mathfrak{G}$ is metrisable by a result of S. Kakutani.[5]

1) K. Yosida: On the group embedded in the metrical complete ring, Jap. J. of Math. **13** (1936). This paper will be cited as [I].

2) The condition γ) in the definition of a *Lie group* in [I] is not an essential one, it states that the group is connected. Hence it may be omitted out of the definition.

3) K. Yosida: On the group embedded in the metrical complete ring, II, to appear soon in Jap. J. of Math. This paper will be cited as [II].

4) For the proof see [I].

5) S. Kakutani: Über die Metrisation der topologischen Gruppen, Proc. **12** (1936). Cf. also G. Birkhoff: A note on topological groups, Comp. Math. **3** (1936).