66. An Extention of the Phragmén-Lindelöf's Theorem.

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Theorem 1. *Let $f(z)$ be a function defined in a domain D, which satisfies the following conditions:*

1°. *$f(z)$ is holomorphic in D.*

2°. *To each point ζ on the boundary C of D with exception of a point z_0, and to each positive number $\varepsilon > 0$, we can associate a circle with the center ζ, in which the following inequality is verified:*

$$|f(z)| \leq m+\varepsilon.$$

3°. *z_0 is a limiting point of the boundary C of D.*

4°. *In a neighbourhood of z_0, $f(z)$ is univalent.*

Then we have $|f(z)| \leq m$ throughout in D.

Proof. Let us describe a circle S with the center z_0; $|z-z_0|=r$ such that $f(z)$ be univalent in the common part of the inside of S and D. Then the domain D is decomposed into at most an enumerable infinity of domains, whose boundaries are contained in the boundary C of D and the circle S. If the following lemma is established, we can see that in each of those domains, $|f(z)|$ is inferior to a fixed constant (valid for all sub-domains), and therefore, $|f(z)|$ is limited in D. Then, applying the Phragmén-Lindelöf's theorem, we can conclude that $|f(z)| \leq m$ throughout in D.

Lemma. *Let $f(z)$ be a function defined in D with the following properties:*

1°. *$f(z)$ is holomorphic and univalent in D.*

2°. *z_0 is a limiting point of the boundary of D.*

3°. *For every frontier point ζ of D distinct from z_0, we have*

$$\overline{\lim_{z\to\zeta}}\, |f(z)| \leq m.$$

Then we have $|f(z)| \leq m$ throughout in D.

Proof of lemma. Let us denote by $\mathfrak{D}$ the set of all the values of $f(z)$, z in D. We shall prove first, that there exist a radius R such that we can not trace any Jordan simple closed curve which contains the circle $|w|=R$ inside, and which is situated in $\mathfrak{D}$.

In fact, suppose that there exists no such radius R, then we have a sequence of Jordan simple closed curves $C_n (n=1, 2, 3, \ldots\ldots)$, in $\mathfrak{D}$, with the following properties:

1) C_n tend uniformly to ∞.

2) C_{n+1} contains C_n inside $(n=1, 2, 3, \ldots\ldots)$.

Then consider the curves Γ_n in D such as C_n is image of Γ_n by means of $f(z)$. Γ_n is any Jordan simple closed curve, and must satisfy the following properties:

1) Γ_n tend uniformly to z_0.