26. The Multiplication-Theorems of the Cauchy Series.

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The chief object of this note is to establish the multiplication-theorems[1] of Cauchy series under the conditions which are in some respects more general than those given in our previous paper. Their connections with certain aspects of an interpolation will also be indicated in § 3.

§ 1. We consider a linear functional $l_\xi\{f(\xi)\}\equiv\int_0^b f(\xi)\,d\varphi(\xi)$ associated to a given function $\varphi(\xi)$ of bounded variations over a finite interval $(0, b)$. To each function $f(x)$, which is Lebesgue-integrable over $(0, b)$, we shall correspond a sequence of contour-integrals[2] defined by

$$(1)\qquad f_r(x)\equiv S_{\mathfrak{C}_r}(x;f)\equiv\frac{1}{2\pi i}\oint_{\mathfrak{C}_r}\frac{e^{\lambda x}}{G(\lambda)}\,l_\xi\left\{e^{\lambda\xi}\int_0^\xi e^{-\lambda\eta}f(\eta)\,d\eta\right\}d\lambda\,,$$

where the integral function $G(\lambda)\equiv l_\xi\{e^{\lambda\xi}\}$.[3] We observe

Theorem I. Consider a function $f(x)$ belonging to $L^p(0, b)$[4] *with $p>1$. If there is a sequence $\{S_{\mathfrak{C}_r}(x;f)\}$ $(r=1, 2, 3, \ldots\ldots)$ such that*

$$(2)\qquad \lim_{r\to\infty}\int_0^b|f(x)-S_{\mathfrak{C}_r}(x;f)|^p\,dx=0\,,$$

then, for any given function $g(x)$ belonging to $L^q(0, b)$, where $1/p+1/q=1$, we have

$$(3)\qquad (f, g)\equiv l_\xi\left\{\int_0^\xi f(\xi-\eta)g(\eta)\,d\eta\right\}$$
$$=\lim_{r\to\infty}\frac{1}{2\pi i}\oint_{\mathfrak{C}_r}\frac{l_\xi\left\{e^{\lambda\xi}\int_0^\xi e^{-\lambda\eta}f(\eta)\,d\eta\right\}l_\xi\left\{e^{\lambda\xi}\int_0^\xi e^{-\lambda\eta}g(\eta)\,d\eta\right\}}{G(\lambda)}\,d\lambda\,.$$

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1) *T. Kitagawa*: On the theory of linear translatable functional equation and Cauchy's series. Jap. Journ. Math., **XIII** (1937) pp. 233–332. Cf. specially Chap. III, §§ 12–14. We shall note this paper by [T].

2) In this note as well as in [T], a sequence of contours $\{\mathfrak{C}_r\}$ $(r=1, 2, 3, \ldots\ldots)$ is always assumed to be selected such that (i) $\mathfrak{C}_r$ is contained in the domain enclosed by $\mathfrak{C}_{r+1}$ and, $\mathfrak{C}_r$ diverges to the whole plane as $r\to\infty$, that is, the distance between the origin and $\mathfrak{C}_r$, which we denote by d_r, tends to infinity as $r\to\infty$; (ii) there are merely two points of the intersections of $\mathfrak{C}_r$ with the imaginary axis of λ-plane, which we denote by $i\rho_r$ and $-i\rho_r$ respectively. The interval $(0, b)$ is assumed to be closed.

3) In [T] we have considered a $\mathfrak{C}$-section of Cauchy series with respect to a linear translatable operation. The present form can be recognised as a special case of the former one, as we can consider a linear translatable operation defined by $l_\xi\{f(x+\xi)\}=\int_0^b f(x+\xi)\,d\varphi(\xi)$. Cf. Also *J. Delsarte*: Les fonctions "moyenne-périodiques." Journ. d. Math. Pures et appliquées, neuvième série, tome quatorziéeme (1935).

4) $L^p(0, b)$ denotes the class of all the functions $f(x)$ which are defined over $(0, b)$ and $|f(x)|^p$ are integrable in the sense of Lebesque.