PAPERS COMMUNICATED

62. *A Proof of a Theorem of Hardy and Littlewood Concerning Strong Summability of Fourier Series.*

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(Comm. by M. FUJIWARA, M.I.A., Oct. 12, 1939.)

1. Let $f(x)$ be integrable and periodic with period 2π and let its Fourier series be

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) . \tag{1}$$

If $f(x) \in L_p$ $(p > 1)$, then (1) is strongly summable for any positive index at a Lebesgue set, that is:

$$\sum_{\nu=0}^{n} | s_\nu(x) - f(x) |^k = o(n) , \tag{2}$$

for every $k > 0$, where s_ν is the partial sums of (1). If $f(x)$ is merely integrable (2) does not necessarily hold at the Lebesgue set.[1] Professors G. H. Hardy and J. E. Littlewood proved, however, the following theorem.[2]

Theorem. If

$$\int_0^t | \phi(u) | \, du = o(t) , \tag{3}$$

then

$$\sum_{\nu=0}^{n} | s_\nu(x) - f(x) |^2 = o(n \log n) , \tag{4}$$

where

$$\phi(u) = \frac{1}{2} \{ f(x+u) + f(x-u) - 2f(x) \} . \tag{5}$$

They proved this theorem by power series method. The object of this paper is to give an elementary proof.

2. We make the ordinary simplifications. Suppose that $f(t)$ is even and $x=0$, $f(0)=0$, so that $\phi(u) = f(u)$. Thus we shall prove, under the condition

$$\int_0^t | f(u) | \, du = \Phi(t) = o(t) , \tag{6}$$

that

$$\sum_{\nu=0}^{n} s_\nu^2 = o(n \log n) . \tag{7}$$

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1) This is due to Hardy and Littlewood, The strong summability of Fourier series, Fund. Math., **25** (1935), 162–189.

2) Hardy-Littlewood, loc. cit. It is unsolved, however, whether (2) holds almost everywhere.