25. *On the Strong Law of Large Numbers.*

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(Comm. by M. FUJIWARA, M.I.A., March 12, 1940.)

1. Let

$$X_1, X_2, \dots, X_n, \dots \tag{1}$$

be a sequence of independent chance variables and let the expectation of X_n, $E(X_n)$ be 0. If

$$\frac{s_n}{n} = \frac{X_1+X_2+\cdots+X_n}{n} \tag{2}$$

converges to zero with probability 1, we say that the sequence (1) obeys to the strong law of large numbers. The ordinary law of large numbers asserts that (2) converges in probability.

Sufficient conditions for the validity of the strong law of large numbers were given by various authors.[1]

Concerning the series of independent chance variables, it is well known that the convergence in probability and the convergence with probability 1 is equivalent. This is due to P. Lévy. Mr. G. Ottaviani has recently given a simple proof[3] of this theorem. For the sequence (2), the similar facts do not necessarily hold. Thus it arises the problem to give the condition to conclude the validity of the strong law from that of the ordinary law of large numbers. The present paper concerns this problem.

2. *Theorem 1. For any positive* ε, *let*

$$Pr\left(\varepsilon > \frac{s_n}{n} > -\varepsilon\right) > 1-\delta_n(\varepsilon), \qquad \delta_n(\varepsilon) \to o \ \text{ as } \ n \to \infty,$$

and suppose that for any $\varepsilon > o$

$$\sum_{k=1}^{\infty} \delta_{2^k}(\varepsilon) < \infty .$$

Then the sequence (1) *converges to zero with probability* 1.

To prove the theorem we use the method of G. Ottaviani. We shall prove the theorem in terms of Lebesgue measure in place of probability and let $X_i = X_i(t)$ $(i=1, 2, \dots)$ are measurable functions independent mutually.

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1) A. Kolmogoroff, Sur la loi forte des grands nombres, Comptes Rendus, **191** (1930), pp. 910–911.

J. Marcinkiewicz—A. Zygmund, Sur les fonctions indépendentes, Fund, Math., **29** (1937).

P. R. Halmos, On a necessary condition for the strong law of large numbers, Ann. Math., **40** (1939).

2) P. Lévy, Théorie de l'addition des variables aléatories (1937), p. 139.

3) G. Ottaviani, Giornali di Mat. (1939). 吉田耕作, 全國紙上數學談話會誌, **188**.