19. *On Some Properties of Umbilical Points of Hypersurfaces.*

By Yosio MUTÔ.
Tokyo Imperial University.

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(1) Let us consider in an $n+1$-dimensional Riemannian space V_{n+1} a hypersurface V_n denoted by

$$x^\lambda = x^\lambda(x^i) \qquad \begin{cases} \lambda, \mu, \nu, \dots = 1, 2, \dots, n+1 \\ i, j, k, \dots = 1, 2, \dots, n. \end{cases}$$

Then we get the following relations:

$$N_\lambda \partial_i x^\lambda = 0, \qquad g_{\lambda\mu} N^\lambda N^\mu = 1,$$

$$H_{jk}^{\cdot\cdot\lambda} = \partial_k \partial_j x^\lambda + \{{}^{\lambda}_{\mu\nu}\} \partial_j x^\mu \partial_k x^\nu - \{{}^{i}_{jk}\} \partial_i x^\lambda = -N_{jk} N^\lambda,$$

where N^λ is the unit vector field normal to V_n and $\{{}^{\lambda}_{\mu\nu}\}$ and $\{{}^{i}_{jk}\}$ are the Christoffel symbols constructed from the fundamental tensors $g_{\mu\nu}$ and g_{jk} of V_{n+1} and V_n respectively. From the second fundamental tensor N_{jk} we can construct the quantity

$$M_{jk} = N_{jk} - \frac{1}{n} g_{jk} N_{lm} g^{lm} \tag{1.1}$$

which is only multiplied by ρ under the transformation $g_{\mu\nu} \to \rho^2 g_{\mu\nu}$.

A line of curvature is a curve $x^i(t)$ which satisfies the equations

$$M^i{}_j \dot{x}^j = \alpha \dot{x}^i. \tag{1.2}$$

When we differentiate (1.2) with respect to t we get

$$M^i{}_j a^j + M^i{}_{jk} \dot{x}^j \dot{x}^k = \alpha a^i + \dot{\alpha} \dot{x}^i, \tag{1.3}$$

$$M^i{}_j b^j + 2M^i{}_{jk} a^j \dot{x}^k + M^i{}_{jk} \dot{x}^j a^k + M^i{}_{jkl} \dot{x}^j \dot{x}^k \dot{x}^l = \alpha b^i + 2\dot{\alpha} a^i + \ddot{\alpha} \dot{x}^i, \tag{1.4}$$

$$\begin{aligned} M^i{}_j c^j &+ 3M^i{}_{jk} b^j \dot{x}^k + M^i{}_{jk} \dot{x}^j b^k + 3M^i{}_{jk} a^j a^k + 3M^i{}_{jkl} a^j \dot{x}^k \dot{x}^l \\ &+ 2M^i{}_{jkl} \dot{x}^j a^k \dot{x}^l + M^i{}_{jkl} \dot{x}^j \dot{x}^k a^l + M^i{}_{jklm} \dot{x}^j \dot{x}^k \dot{x}^l \dot{x}^m \\ &= \alpha c^i + 3\dot{\alpha} b^i + 3\ddot{\alpha} a^i + \dddot{\alpha} \dot{x}^i, \end{aligned} \tag{1.5}$$

where

$$\begin{aligned} a^i &= \ddot{x}^i + \{{}^{i}_{jk}\} \dot{x}^j \dot{x}^k \\ b^i &= \dot{a}^i + \{{}^{i}_{jk}\} a^j \dot{x}^k \\ c^i &= \dot{b}^i + \{{}^{i}_{jk}\} b^j \dot{x}^k \end{aligned} \tag{1.6}$$

and $M^i{}_{jk}$, $M^i{}_{jkl}$ etc. are the covariant derivatives of $M^i{}_j$ with respect to g_{jk}.

We call a point on the V_n a perfectly umbilical point when there is a line of curvature passing through the point in each direction.