17. *Some Theorems on Abstractly-valued Functions in an Abstract Space.*

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1. *Introduction and Theorems.* Let $f(t)$ be an abstractly-valued function defined on $[0, 1]$ whose range lies in a Banach space $\mathfrak{X}$. Under $L^p(\mathfrak{X})\ (p \geqq 1)\ \big(L^1(\mathfrak{X}) = L(\mathfrak{X})\big)$ we understand the class of all functions $f(t)$ measurable in the sense of S. Bochner such that $\int_0^1 \|f(t)\|^p dt < \infty$. $L^p(\mathfrak{X})\ (p \geqq 1)$ is a Banach space with $\|f\| = \left(\int_0^1 \|f(t)\|^p dt\right)^{\frac{1}{p}}$ as its norm.

The purpose of the present note is to prove the following theorems:

Theorem 1. In an arbitrary space T let $\mathfrak{E}$ be a Borel family of subsets that includes T, and $\alpha(E)$ be a non-negative set function which is completely additive over $\mathfrak{E}$. If an abstractly-valued function $X(E)$, defined from $\mathfrak{E}$ to a Banach space $\mathfrak{X}$, is weakly absolutely continuous (i.e., for each φ in $\bar{\mathfrak{X}}$, the numerical function $\varphi X(E)$ is completely additive and absolutely continuous), then $X(E)$ is even strongly absolutely continuous (i.e., $X(E)$ is strongly completely additive, and for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|X(E)\| < \varepsilon$ whenever $\alpha(E) < \delta$).

Theorem 2. If $\mathfrak{X}$ is locally weakly compact, and if a sequence $\{f_n(t)\}$ $(n=1, 2, \dots)$ of elements of $L(\mathfrak{X})$ is equi-integrable, then $\{f_n(t)\}$ $(n=1, 2, \dots)$ contains a subsequence which converges weakly (as a sequence in $L(\mathfrak{X})$) to an element $f(t) \in L(\mathfrak{X})$.

Theorem 3. If $\mathfrak{X}$ is locally weakly compact, then $L^p(\mathfrak{X})\ (p > 1)$ is also locally weakly compact.

Theorem 4. If $\mathfrak{X}$ is locally weakly compact, then $L(\mathfrak{X})$ is weakly complete.

Theorem 4 is a generalization of a result of S. Bochner-A. E. Taylor,[1] who assumed that $\mathfrak{X}$ is reflexive and that $\mathfrak{X}$ and $\bar{\mathfrak{X}}$ both satisfy the condition (D). Theorem 2[2] is an analogue of H. Lebesgue's theorem,[3] which is concerned with numerical-valued functions. These two theorems will be proved by using Theorem 1, and this theorem was announced without proof by B. J. Pettis[4] under the additional assumption[5] that T is expressible in the form: $T = \sum_{i=1}^{\infty} T_i$ with $\alpha(T_i) < \infty$, $i = 1, 2, \dots$.

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1) S. Bochner-A. E. Taylor: Linear functionals on certain spaces of abstractly-valued functions, Annals of Math., **39** (1938), 913–944. Theorem 5.2.

2) Theorem 2 may be considered as a precision to Theorem 4.2. (p. 923) in the paper of S. Bochner-A. E. Taylor cited in (1).

3) H. Lebesgue: Sur les intégrales singulières, Ann. de la Fac. des Sci. de Toulose, **1** (1909), especially p. 52.

4) B. J. Pettis: Bull. Amer. Math. Soc., (Abstracts), **44**-**2** (1939), 677.

5) This fact was suggested to me by K. Yosida.