16. *Weak Topology, Bicompact Set and the Principle of Duality.*

By Shizuo KAKUTANI.
Mathematical Institute, Osaka Imperial University.
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1. *Weak topology and bicompact set.* Let $\mathfrak{S}$ be an abstract space and consider the family Ω of real-valued (bounded) functions $f(x)$ defined on $\mathfrak{S}$. We shall introduce a weak topology on Ω. For any $f_0(x)\in\Omega$ its weak neighbourhood $U(f_0; x_1, x_2, \dots, x_n; \varepsilon)$ is defined as the totality of all the functions $f(x)\in\Omega$ such that $|f(x_i)-f_0(x_i)|<\varepsilon$ for $i=1, 2, \dots, n$, where $\{x_i\}(i=1, 2, \dots, n)$ is an arbitrary finite system of points from $\mathfrak{S}$, and $\varepsilon>0$ is an arbitrary positive number. It is clear that this neighbourhood system defines a topology on Ω. This topology is called the *$\mathfrak{S}$-weak topology of Ω as functionals.*

In the same way, if we consider, for any $x_0\in\mathfrak{S}$, $x_0(f)\equiv f(x_0)$ as a real-valued (bounded) function defined on Ω, we can introduce a weak topology on $\mathfrak{S}$. Indeed, for any $x_0\in\mathfrak{S}$ its weak neighbourhood $U(x_0; f_1, f_2, \dots, f_n; \varepsilon)$ is defined as the totality of all the points $x\in\mathfrak{S}$ such that $|f_i(x)-f_i(x_0)|<\varepsilon$ for $i=1, 2, \dots, n$, where $\{f_i(x)\}(i=1, 2, \dots, n)$ is an arbitrary finite system of functions from Ω, and $\varepsilon>0$ is an arbitrary positive number. It is again clear that this neighbourhood system defines a topology on $\mathfrak{S}$. This topology is called the *Ω-weak topology of $\mathfrak{S}$ as points.*

Theorem 1. If Ω is the totality of all the bounded (not necessarily continuous) real-valued functions $f(x)$ defined on $\mathfrak{S}$ such that $|f(x)|\leqq 1$ for any $x\in\mathfrak{S}$, then $\mathfrak{S}$ is bicompact with respect to the $\mathfrak{S}$-weak topology of Ω as functionals.

This theorem is due to A. Tychonoff.[1] The weak topologies of the same kind may also be defined analogously even if the range of $f(x)$ is contained in an arbitrary uniform space (in the sense of A. Weil).[2] In the special case, when $\mathfrak{S}$ is a Banach space E and Ω is the set of all the bounded linear functionals $f(x)$ defined on E (i. e., $\Omega=\bar{E}$), these weak topologies become the usual ones. We have once[3] studied the weak topologies of Banach spaces, and have shown that these weak topologies are useful in the problems concerning the regularity of Banach spaces. From Theorem 1 we can easily deduce

Theorem 2. Let E be a Banach space. Then the unit sphere: $\|f\|\leqq 1$ of the conjugate space $\bar{E}$ of E is bicompact with respect to the E-weak topology of $\bar{E}$ as functionals.

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1) A. Tychonoff: Über einen Funktionenraum, Math. Ann., **111** (1935), 762–766.

2) A. Weil: Sur les espaces à structure uniforme et sur la topologie générale, Actualité, 551, Paris, 1937.

3) S. Kakutani: Weak topology and regularity of Banach spaces, Proc. **15** (1939), 169–173.